



(F, L) Contractions on Complete Weak Partial Metric Spaces for a Commuting Family of Self Mappings

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

The aim of this paper is to clarify the choice of the self map $T : X \rightarrow X$ in Kaya et als (F, L) weak contractions by choosing a family $T_n, n \in \mathbb{N}$ of (F, L) contractions. Motivated by the fact that the uniform limit T of the family of self maps is a better approximation, we are guaranteed the choice of the self map. By this, the choice of T is no longer arbitrary. Again, for any finite family $T_1, T_2, T_3, \dots, T_N$ of (F, L) contractions their composition is an (F, L) contraction. This concept generalizes and improves on several results especially Theorems 3.1 and 3.2 of [10].

Keywords: (F, L) contractions; fixed point; common fixed point; weak partial metric space; commuting family of self maps.

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1 Introduction

The theory of metric spaces has experienced rapid development in recent decades and has gained access to many areas of Mathematics such as group theory, Riemann geometry and partial differential equations. Metric space methods have also been employed in internet search engines and image classification [1] as well as protein classification [2].

In recent years the attention in this area of research has shifted to partial metric spaces. The concept of partial metric was introduced by S. G. Matthews [3] in 1994 as a generalization of metric spaces. Matthews [3] replaced the equality $d(x, x) = 0$ in the definition of a metric with the inequality $d(x, x) \leq d(x, y)$ for all $x, y \in X$. He then proved Banachs contraction mapping principle in a new framework and also discussed some properties of convergence of sequences.

In [4], Oltra and Valero generalized the results of Matthews, and Altun et al [5] improved on Matthews results by studying generalized contractions on this space. Again, Romaguera [6] characterized the completeness for partial metric spaces. Heckmann [7] omitted the self-distance axiom of partial metric and defined and pioneered the theory of weak-partial metrics. Wardowski [8] used the notion of F-contractions to prove fixed point theorems which generalized Banachs contraction mapping principle. Later, Wardowski [9] working in tandem with Van Dung introduced F-weak contractions and established fixed point theorems for such mappings in a complete metric spaces. These theorems extend and generalize several theorems in this direction and improve many existing results.

In 2017, Kaya et al [10] defined the concept of (F, L) contractions in weak partial metric spaces and proved some common fixed point results for a self mapping T. The motivation for the concept of a commuting family of (F, L) contractions stems from the work of Frimpong and Prempeh [11]. They obtained fixed point results in 2017 in reflexive Banach space using a family of maps.

In this paper, my purpose is to prove some common fixed point results for a commuting family $T_n, n = 1, 2, \dots$ of self mappings in a weak partial metric space. This generalizes and improves several recent results: particularly Theorems 3.1 and 3.2 of [10].

Particularly, if $T_n, n = 1, 2 \dots N$ is a sequence of (F,L) contractions in a complete partial metric space then their composition $T = T_1 T_2 T_3 \dots T_N$ is an (F,L) contraction.

2 Preliminaries

We now mention briefly some fundamental definitions and results which will be needed subsequently.

Definition 2.1 [3]

A metric on a non empty set E is a function $d : E \times E \rightarrow \mathbb{R}$ satisfying the following axioms:

$$M_1 : d(x, y) \geq 0, \quad \forall \text{ pair } x, y \in E$$

$$M_2 : d(x, y) = 0 \Leftrightarrow x = y \text{ for all } x, y \in E$$

$$M_3 : d(x, y) = d(y, x) \quad \forall x, y \in E$$

$$M_4 : d(x, z) \leq d(x, y) + d(y, z) \text{ for all } x, y, z \in E.$$

When d is a metric on E then the pair (E, d) is called a metric space.

Definition 2.2 [3]

A partial metric on a non-empty set E is a function $\rho : E \times E \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in E$:

$$P_1 : x = y \Leftrightarrow \rho(x, x) = \rho(x, y) = \rho(y, y)$$

$$P_2 : \rho(x, x) \leq \rho(x, y)$$

$$P_3 : \rho(x, y) \leq \rho(y, x)$$

$$P_4 : \rho(x, z) \leq \rho(x, y) + \rho(y, z) - \rho(y, y)$$

A partial metric space is a pair (E, ρ) such that E is a non-empty set and ρ is a partial metric on E . If E is a partial metric space then each metric ρ on E generates a T_0 topology λ_ρ on E with a base consisting of the family of open balls $\{B_\rho(x, \varepsilon) : x \in E, \varepsilon > 0\}$, where $B_\rho(x, \varepsilon) = \{y \in E : \rho(x, y) < \rho(x, x) + \varepsilon\}$, for all $x \in E$ and $\varepsilon > 0$. Kaya et al [10]

Definition 2.3 [3]

Let (E, ρ) be a partial metric space. Then,

- a sequence $\{\mu_n\}$ in (E, ρ) converges with respect to λ_ρ to a point $\mu \in E$ if $\rho(\mu, \mu) = \lim_{n \rightarrow \infty} \rho(\mu, \mu_n)$,
- a sequence $\{\mu_n\}$ in (E, ρ) is called Cauchy if $\lim_{m, n \rightarrow \infty} \rho(\mu_m, \mu_n)$ exists and is finite,
- (E, ρ) is said to be complete if every Cauchy sequence $\{\mu_n\}$ in (E, ρ) converges to a point $\mu \in E$; that is

$$\rho(\mu, \mu) = \lim_{m, n \rightarrow \infty} \rho(\mu_m, \mu_n)$$

Lemma 2.1 [10]

Let ρ be a partial metric as defined in 2.3 above. Then the function $d : E \times E \rightarrow \mathbb{R}^+$ defined by

$$d(x, y) = \rho(x, y) - \min\{\rho(x, x), \rho(y, y)\}$$

\forall pair $x, y \in E$ is an ordinary metric on E .

Lemma 2.2 [3]

Let (E, ρ) be a partial metric space. Then a sequence $\{\mu_n\}$ in (E, ρ) is Cauchy if and only if it is a Cauchy sequence in the metric space (E, d) . Moreover, (E, ρ) is complete if and only if (E, d) is complete. Additionally,

$$\lim_{n \rightarrow \infty} d(\mu_n, \mu) = 0 \Leftrightarrow \rho(\mu, \mu) = \lim_{n \rightarrow \infty} \rho(\mu_n, \mu) = \lim_{n, m \rightarrow \infty} \rho(\mu_n, \mu_m)$$

Definition 2.4 [10]

Let (E, ρ) be a partial metric space. A mapping $T : E \rightarrow E$ is said to be continuous at a point $x_0 \in E$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $T(B_\rho(x_0, \delta)) \subset B_\rho(Tx_0, \varepsilon)$. We say that T is continuous on (E, ρ) if it is continuous at all points $x \in E$.

Lemma 2.3 [10]

Let (E, ρ) be a complete partial metric space, $T : E \rightarrow E$ a continuous map and $\{\mu_n\}$ a sequence in E such that $\mu_n \rightarrow \mu \in E$, then $\lim_{n \rightarrow \infty} \rho(T\mu_n, T\mu) = \rho(T\mu, T\mu)$.

Definition 2.5 [8]

Let \mathcal{F} be the family of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following axioms:

- F_1 : F is a strictly increasing function. i.e. for all $\alpha, \beta \in (0, \infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$,
- F_2 : for each sequence $\{\mu_n\}$ of positive integers in $(0, \infty)$, the following holds:

$$\lim_{n \rightarrow \infty} \mu_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} F(\mu_n) = -\infty.$$

- F_3 : $\exists k \in (0, 1) \ni \lim_{\alpha \rightarrow 0^+} \{\alpha^k F(\alpha)\} = 0$.

Definition 2.6 [9]

Let (E, d) be a metric space. A mapping $T : E \rightarrow E$ is said to be an F -contraction on (E, d) if $\exists F \in \mathcal{F}$ and $\tau > 0$ such that for every pair $x, y \in E$ satisfying $d(Tx, Ty) > 0$, the following condition holds:

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Definition 2.7 [10]

A weak partial metric on a non-empty set E is a function $\rho : E \times E \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in E$:

- K_1 : $x = y \Leftrightarrow \rho(x, x) = \rho(x, y) = \rho(y, y)$
- K_2 : $\rho(x, y) \leq \rho(y, x)$
- K_3 : $\rho(x, z) \leq \rho(x, y) + \rho(y, z) - \rho(y, y)$

Definition 2.8 [10]

Let T_1 and T_2 be self mappings on E . If $T_1x = T_2x = y$ for some $x \in E$ then x is called a coincidence point of T_1 and T_2 and y is called a point of coincidence of T_1 and T_2 . The self-mappings T_1 and T_2 are said to be weakly compatible if they commute at their coincidence point. If T_1 and T_2 have a unique point of coincidence then this point is their unique common fixed point.

When ρ is a weak partial metric on E then the pair (E, ρ) is called a weak partial metric space. The concept of weak partial metric space was introduced by Heckmann [7] as a generalization of Matthews [3] partial metric space by omitting the small self-distance axiom of Matthews definition. Heckmann then introduced the weak small self-distance axiom: that is, for every pair $x, y \in E$,

$$\rho(x, y) \geq \frac{\rho(x, x) + \rho(y, y)}{2}$$

With this property Heckmann was able to show that weak partial metric spaces are no different from small self-distance axiom. His conclusion was that every partial metric space is a weak partial metric space but the converse may not be true.

3 Main Results

We now discuss our results by introducing the following concept.

Definition 3.1 [10]

Let (E, ρ) be a weak partial metric space. A mapping $T : E \rightarrow E$ is said to be an F - contraction on (E, ρ) if $\exists F \in \mathcal{F}$ and $\tau > 0$ such that for every pair $x, y \in E$ satisfying $\rho(Tx, Ty) > 0$, the following holds:

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)) \tag{3.1}$$

Definition 3.2 [10]

Let (E, ρ) be a weak partial metric space. A mapping $T : E \rightarrow E$ is said to be an (F, L) - contraction on (E, ρ) if $F \in \mathcal{F}$ and $\exists \tau > 0$ and $L \geq 0$ for every pair $x, y \in E$ satisfying $\rho(Tx, Ty) > 0$, the following holds:

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)) + Ld(y, Tx). \tag{3.2}$$

By virtue of the symmetry of the metric, the (F, L) - contraction condition implicitly includes the dual

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)) + Ld(x, Ty). \tag{3.3}$$

Using (3.2) and (3.3), the (F, L) - contraction condition can be replaced by

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)) + L \min\{d(x, Ty), d(y, Tx)\}. \tag{3.4}$$

Lemma 3.1 [10]

Let (E, ρ) be a weak partial metric space and $T : E \rightarrow E$ a self mapping. T is said to be an F - contraction on (E, ρ) if $\exists F \in \mathcal{F}$ and $\tau > 0$ such that for every pair $x, y \in E$ satisfying $\rho(Tx, Ty) > 0$, the following holds: $\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y))$. By virtue of the fact that

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)) \leq F(\rho(x, y)) + L \min\{d(x, Ty), d(y, Tx)\},$$

T is an (F, L) - contraction. Thus, every F - contraction is an (F, L) - contraction.

Lemma 3.2 [10]

Let (E, ρ) be a weak partial metric space and $T : E \rightarrow E$ a self mapping. Suppose that $F \in \mathcal{F}$ and $\exists \tau > 0$ and $L \geq 0$ for every pair $x, y \in E$ satisfying $\rho(Tx, Ty) > 0$, the following holds:

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y)) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}.$$

Theorem 3.1

Let (E, ρ) be a complete weak partial metric space and $T_n : E \rightarrow E$, $n = 1, 2, 3, \dots, \dots, N$ be a sequence of commuting (F, L) contractions on E . Then their composition $T = T_1 T_2 T_3 \dots T_N$ is (F, L) contraction on E .

Proof. The proof is by induction. Let

$$P_N : \tau + F\{\rho(T_1 T_2 T_3 \dots T_N(x), T_1 T_2 T_3 \dots T_N(y))\} \leq F\{\rho(x, y)\} + L \min\{d(x, Ty), d(y, Tx)\}$$

P_1 is trivially true by hypothesis.

$$P_2 : \tau + F\{\rho(T_2 T_1 x, T_2 T_1 y)\} \leq F\{\rho(T_1 x, T_1 y)\} \leq F\{\rho(x, y)\} + L \min\{d(x, T_1 y), d(y, T_1 x)\}$$

$$\implies \tau + F\{\rho(T_2 T_1 x, T_2 T_1 y)\} = \tau + F\{\rho(T_1 T_2 x, T_1 T_2 y)\} \leq F\{\rho(x, y)\} + L \min\{d(x, T_2 y), d(y, T_2 x)\}.$$

Hence T_2 is (F, L) contraction on E .

Now, assuming that for some $k \in \mathbb{N}$, T_1, T_2, \dots, T_{k-1} are (F, L) contraction on E , we get

$$\tau + F\{\rho(T_k T_{k-1} \dots T_1(x), T_k T_{k-1} \dots T_1(y))\} \leq F\{\rho(x, y)\} + L \min\{d(x, T_k y), d(y, T_k x)\}.$$

Therefore $T = T_1 T_2 T_3 \dots T_N$ is (F, L) contraction on E . □

Theorem 3.2 [11]

Let E be a weak partial metric space and let C be a bounded closed convex subset of E . Let T_n , $n \geq 1$ be a sequence of (F, L) contractions on C such that $T_n(x) \leq T_{n+1}(x)$, $\forall n \geq 1, x \in E$. If T_n converges pointwise on C to (F, L) contraction T then the convergence is uniform.

Proof. Let $f_n(x) = T(x) - T_n(x)$ for each $n \in \mathbb{N}$. Then f_n is a sequence of (F, L) contractions on the compact set C such that $f_n(x) \geq f_{n+1}(x) \geq 0$ for all $x \in C$ and $n \in \mathbb{N}$.

Moreover,

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \{T(x) - T_n(x)\} = 0$$

Let $M_n = \sup\{f_n(x) : x \in C\}$ and let $\varepsilon > 0$ be given.

Let

$$E_n = \{x \in C : f_n(x) < \varepsilon\} = f_n^{-1}((-\infty, \varepsilon)).$$

Then E_n is open for each n and $E_n \subset E_{n+1}$ because $f_n(x) \geq f_{n+1}(x)$. Since for each $x \in C$, $\lim_{n \rightarrow \infty} f_n(x) = 0$, $\exists n \in \mathbb{N} \ni f_n(x) < \varepsilon$ which implies $x \in E_n$. Thus $\cup_{n=1}^{\infty} E_n$ is an open cover for C and $\cup_{n=1}^{\infty} E_n = C$.

Since C is compact there exists a finite subcover for C and in view of the fact that $E_n \subset E_{n+1}$, the largest of these also covers C . Hence there is $N \in \mathbb{N}$ such that $E_N = C$ and this means that $f_N(x) < \varepsilon$ for all $x \in C$ and $n \geq N$. Thus $M_N \leq \varepsilon$ and since $M_n \geq 0$, $\lim_{n \rightarrow \infty} M_n = 0$. This indicates that the sequence f_n converges uniformly to 0 on C and therefore the sequence of (F, L) contractions T_n converges uniformly to T on C . □

Theorem 3.3

Let (E, ρ) be a complete weak partial metric space and $T_n : E \rightarrow E$, $n = 1, 2, 3, \dots, \dots, N$ be a sequence of commuting (F, L) contractions on E . If F is continuous then $T = T_1 T_2 T_3 \dots T_N$ has a unique fixed point in E .

Proof. Let $\mu_{n+1} = T\mu_n$, $n \in \mathbb{N}$ be a sequence in E , where $\mu_0 \in E$ is an arbitrary point.

Case 1: If $\mu_n = \mu_{n+1}$ for some $n \in \mathbb{N}$, then μ_n is a fixed point of $T = T_1 T_2 T_3 \dots T_N$ and the proof is complete.

Case 2: Suppose that $\mu_n \neq \mu_{n+1} \forall n \in \mathbb{N}$. Then we get using equation (4)

$$\begin{aligned} F\{\rho(\mu_n, \mu_{n+1})\} &= F\{\rho(T\mu_{n-1}, T\mu_n)\} \\ &\leq F\{\rho(\mu_{n-1}, \mu_n)\} + L \min\{d(\mu_{n-1}, T\mu_n), d(\mu_n, T\mu_{n-1})\} - \tau \\ &= F\{\rho(\mu_{n-1}, \mu_n)\} + L \min\{d(\mu_{n-1}, \mu_{n+1}), d(\mu_n, \mu_n)\} - \tau \\ &= F\{\rho(\mu_{n-1}, \mu_n)\} - \tau \\ \implies F\{\rho(\mu_n, \mu_{n+1})\} &\leq F\{\rho(\mu_{n-1}, \mu_n)\} - \tau \end{aligned} \quad (\star)$$

After n iterations of (\star) we get

$$F\{\rho(\mu_n, \mu_{n+1})\} \leq F\{\rho(\mu_0, \mu_1)\} - n\tau, \quad \forall n \in \mathbb{N} \quad (3.5)$$

Taking limit of (3.5) as $n \rightarrow \infty$ we get

$$\lim_{n \rightarrow \infty} F\{\rho(\mu_n, \mu_{n+1})\} = -\infty$$

which together with F_2 gives

$$\lim_{n \rightarrow \infty} \rho(\mu_n, \mu_{n+1}) = 0 \quad (3.6)$$

Now, by definition of $F \in \mathcal{F}$, $\exists k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} \{\rho(\mu_n, \mu_{n+1})\}^k F\{\rho(\mu_n, \mu_{n+1})\} = 0 \quad (3.7)$$

Combining (3.5) and (3.7) and taking limit as $n \rightarrow \infty$ gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \{\rho(\mu_n, \mu_{n+1})\}^k \{F\{\rho(\mu_n, \mu_{n+1})\} - F\{\rho(\mu_0, \mu_1)\}\} &\leq \lim_{n \rightarrow \infty} \{\rho(\mu_n, \mu_{n+1})\}^k n\tau \leq 0 \\ \implies 0 &\leq \lim_{n \rightarrow \infty} \{\rho(\mu_n, \mu_{n+1})\}^k n\tau \leq 0 \\ \implies \lim_{n \rightarrow \infty} \{\rho(\mu_n, \mu_{n+1})\}^k n &= 0 \end{aligned} \quad (3.8)$$

From equation (3.8) there exists a positive integer φ_1 such that $\forall n \geq \varphi_1$

$$\begin{aligned} \{\rho(\mu_n, \mu_{n+1})\}^k n &\leq 1 \\ \implies \rho(\mu_n, \mu_{n+1}) &\leq \frac{1}{n^{\frac{1}{k}}}, \quad \forall n \geq \varphi_1 \end{aligned} \quad (3.9)$$

Now, let $m, n \in \mathbb{N} \ni n > m \geq \varphi_1$. Using (3.9) and P_3 gives,

$$\begin{aligned} \rho(\mu_n, \mu_m) &\leq \rho(\mu_n, \mu_{n+1}) + \rho(\mu_{n+1}, \mu_{n+2}) + \rho(\mu_{n+2}, \mu_{n+3}) + \cdots + \rho(\mu_{m-1}, \mu_m) - \\ &\quad \{\rho(\mu_{n+1}, \mu_{n+1}) + \rho(\mu_{n+2}, \mu_{n+2}) + \rho(\mu_{n+3}, \mu_{n+3}) + \cdots + \rho(\mu_{m-1}, \mu_{m-1})\} \\ &\leq \sum_{\zeta=n}^{m-1} \rho(\mu_\zeta, \mu_{\zeta+1}) \\ &\leq \sum_{\zeta=n}^{\infty} \rho(\mu_\zeta, \mu_{\zeta+1}) \\ &\leq \sum_{\zeta=n}^{\infty} \frac{1}{\zeta^{\frac{1}{k}}} \end{aligned}$$

Since $k \in (0, 1)$, the series $\sum_{\zeta=n}^{\infty} \frac{1}{\zeta^{\frac{1}{k}}}$ is a p -series with $p > 1$. Therefore $\sum_{\zeta=n}^{\infty} \frac{1}{\zeta^{\frac{1}{k}}} < \infty$ which implies that $\lim_{m,n \rightarrow \infty} \rho(\mu_n, \mu_m) = 0$, and the sequence $\{\mu_n\}$ is Cauchy in (E, ρ) . Then Lemma 2.2 guarantees that $\{\mu_n\}$ is a Cauchy sequence in (E, d) . Moreover, the completeness of (E, ρ) ensures that (E, d) is also complete. Hence there exists $\mu^* \in E$ such that

$$\lim_{n \rightarrow \infty} d(\mu_n, \mu^*) = 0$$

Assume that F is continuous. Then there exist a positive integer $n_1 \ni \forall n \geq n_1 \rho(\mu_{n+1}, T\mu^*) > 0$. Therefore by the definition of our (F-L) contraction in (4), we get

$$\begin{aligned} \tau + F\{\rho(\mu_{n+1}, T\mu^*)\} &= \tau + F\{\rho(T\mu_n, T\mu^*)\} \\ &\leq F\{\rho(\mu_n, \mu^*)\} + L \min\{d(\mu_n, T\mu^*), d(\mu^*, T\mu_n)\} \\ &= F\{\rho(\mu_n, \mu^*)\} + L \min\{d(\mu_n, T\mu^*), d(\mu^*, \mu_{n+1})\} \end{aligned}$$

Again from Lemma 2.2, there exists a positive integer $n_2 \ni \forall n \geq n_2, \tau + F\{\rho(\mu_{n+1}, T\mu^*)\} \leq F\{\rho(\mu^*, \mu^*)\}$ and we get $\rho(\mu_n, \mu^*) < \rho(\mu^*, T\mu^*)$. Thus $\forall n \geq \max\{n_1, n_2\}$,

$$\tau + F\{\rho(\mu_{n+1}, T\mu^*)\} \leq F\{\rho(\mu^*, \mu^*)\} + L \min\{d(\mu_n, T\mu^*), d(\mu^*, \mu_{n+1})\}$$

By the fact that F is continuous and letting $n \rightarrow \infty$, we arrive at

$$\tau + F\{\rho(\mu^*, T\mu^*)\} \leq F\{\rho(\mu^*, \mu^*)\} \tag{3.10}$$

(10) is clearly a contradiction since $\tau > 0$. Hence our claim that $\rho(\mu^*, T\mu^*) > 0$ is untenable. Therefore we must have $\rho(\mu^*, T\mu^*) = 0$. Hence by axioms P_1 and P_2 of Definition 2.2, $T\mu^* = \mu^*$ proving that μ^* is a fixed point of T .

For uniqueness of μ^* , we assume that μ_1^* and μ_2^* are two distinct fixed points of T . Then $\rho(T\mu_1^*, T\mu_2^*) > 0$. By Lemma 3.2 we get

$$\begin{aligned} \tau + F\{\rho(\mu_1^*, \mu_2^*)\} &= \tau + F\{\rho(T\mu_1^*, T\mu_2^*)\} \\ &\leq F\{\rho(\mu_1^*, \mu_2^*)\} + L \min\{d(\mu_1^*, T\mu_1^*), d(\mu_1^*, T\mu_2^*), d(\mu_2^*, T\mu_1^*)\} \\ &= F\{\rho(\mu_1^*, \mu_2^*)\} \end{aligned}$$

This is a contradiction. Thus we have $\mu_1^* = \mu_2^*$, proving uniqueness of fixed point of T . □

Lemma 3.3 [11]

Let E be a non-empty set. Then all equivalent classes of E are disjoint and E is the union of its equivalence classes.

Theorem 3.4

Let (E, ρ) be a complete weak partial metric space and let $T_n : E \rightarrow E, n = 1, 2, 3, \dots$ be an infinite family of commuting (F, L) contractions on E . If F is continuous then $T = T_1 T_2 T_3 \dots T_N$, where $T_1, T_2, T_3, \dots T_N$ is an equivalence class of size N has a unique fixed point in E .

Proof. The proof is done by partitioning the infinite family $T_n, n = 1, 2, 3, \dots$ into equivalence classes of size N , for a positive integer N . Thus, all the assumptions made for the finite number of non-expansive maps in Theorem 3.3 hold for each class. Again, since equivalent classes are disjoint, there are no spillovers into other classes. Therefore any result that is true for one class will also hold for other classes. Consequently, the result is a priori true from Theorem 3.1. □

4 Conclusion

Kaya et al [10] considered in a weak partial metric space X a single self mapping T and proved that if the real-valued function $F : (0, \infty) \rightarrow \mathbb{R}$ as defined in Definition 2.5 is continuous on its domain then T has a unique fixed point in X . The choice of the self mapping T is arbitrary. Thus we have considered a family $T_n : E \rightarrow E$, $n = 1, 2, 3, \dots$, of commuting (F, L) contractions in a weak partial space E .

Case 1 If a finite number of the commuting family is considered then its composition as proved in Theorem 3.1 is (F, L) contraction. Thus the choice of the map is now not arbitrary.

Case 2 For an infinite family of commuting (F, L) contractions, the uniform limit as demonstrated in Theorem 3.2 provides a better approximation for the self map T . Thus, the principal conclusion from the research is that the choice of the self map has been made explicitly clear to be the uniform limit of the sequence of (F, L) contractions.

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Competing Interests

Author has declared that no competing interests exist.

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