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Inverse Analogue of Ailamujia Distribution with Statistical Properties and Applications

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Authors' contributions

This work carried out in collaboration among all authors. Author A. Aijaz designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors A. Ahmad and RT managed the analysis of the study. Author A. Aijaz managed the literature searches. All authors read and approved the final manuscript.

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Abstract

The present paper deals with the inverse analogue of Ailamujia distribution (IAD). Several statistical properties of the newly developed distribution has been discussed such as moments, moment generating function, survival measures, order statistics, shanon entropy, mode and median .The behavior of probability density function (p.d.f) and cumulative distribution function (c.d.f) are illustrated through graphs. The parameter of the newly developed distribution has been estimated by the well known method of maximum likelihood estimation. The importance of the established distribution has been shown through two real life data.

Keywords: Ailamujia distribution; inverse approach; moments; survival analysis; maximum likelihood estimation.

Mathematics subject classification: 60E05; 62E15.

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1 Introduction

H.Q.LV et al. [1] has been established new distribution for modeling real life data and named it Ailamujia distribution with following probability density function and cumulative distribution function.

$$f(x,\alpha) = 4\alpha^2\, x\, e^{-2\alpha x}\,;x > 0\,,\alpha > 0 \tag{1.1}$$

The corresponding cumulative distribution function is given as

$$F(x,\alpha) = 1 - (1+2\alpha x)e^{-2\alpha x}\,,\alpha > 0, x > 0 \tag{1.2}$$

They have studied its various statistical and mathematical properties such as mean, variance, median and maximum likelihood estimation. Ailamujia distribution is a functional distribution used to model the repair time and provide assurance to distribution delay time. Recently several authors have done lot of work on Ailamujia distribution. Pan et al. [2] has worked on Ailamujia distribution for interval estimation and hypothesis testing based on small sample size. Long [3] has obtained its Bayesian estimation under type II censoring on the basis of conjugate prior, Jeffrey's prior and no informative prior distribution. Yu et al. [4] proposed a new method by applying Ailamujia distribution to solve the problem in the production and distribution of battle field injury in campaign macrocosm. Uzma et al. [5] has introduced the weighted analogue of Ailamujia distribution and studied its various characteristics. They showed that weighted analogue of Ailamujia distribution performs better than Ailamujia distribution. Ahmad et al. [6] proposed and studied several structural properties of weighted analogue of inverse Maxwell distribution. Rather et al. [7] proposed a size biased Ailamujia distribution and apply it for analyzing data from engineering and medical science. B.Jaya Kumar et al. [8] introduced area biased distribution and have been studied its various properties and applied the obtained distribution to bladder cancer data. Recently Ahmad et al. [9] introduced two parameter distribution named it Hamza distribution and studied its different mathematical properties.

2 The Inverse Ailamujia Distribution

If X be a random variable follows Ailamujia distribution having p.d.f (1.1), then $Y = \frac{1}{X}$ is said to follow inverse Ailamujia distribution (IAD) if its cumulative distribution function (c.d.f) is defined as

$$F(Y) = P(Y \le y) = P\left(\frac{1}{X} \le y\right)$$

$$= P\left(\frac{1}{y} \le X\right) = P\left(X \ge \frac{1}{y}\right) = 1 - P\left(X < \frac{1}{y}\right) = 1 - F\left(\frac{1}{y}\right)$$

$$F(Y) = \frac{(2\alpha + y)}{y} e^{-\frac{2\alpha}{y}} \quad ,y > 0\,,\alpha > 0 \tag{2.1}$$

The corresponding probability distribution function (p.d.f) is

$$f(y,\alpha) = 4\alpha^2 \frac{1}{y^3} e^{-\frac{2\alpha}{y}} \quad ,y > 0\,,\alpha > 0 \tag{2.2}$$

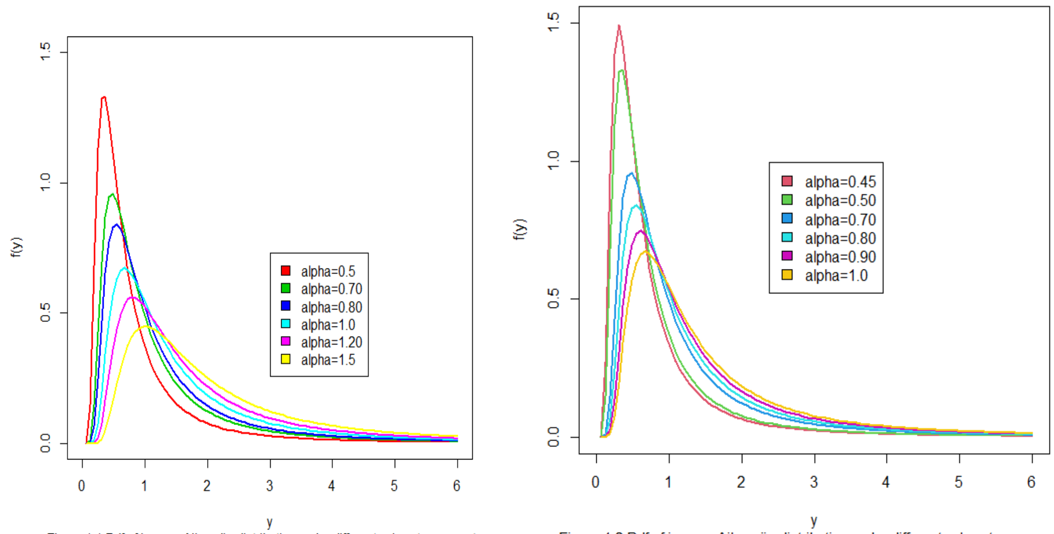


Figure 1.1:Pdf of inverse Ailamujia distribution under different values to parameters

Figure 1.2:Pdf of inverse Ailamujia distribution under different values to parameter

Figs. 1.1 and 1.2. Illustrates different shapes of P.d.f of inverse Ailamujia distribution

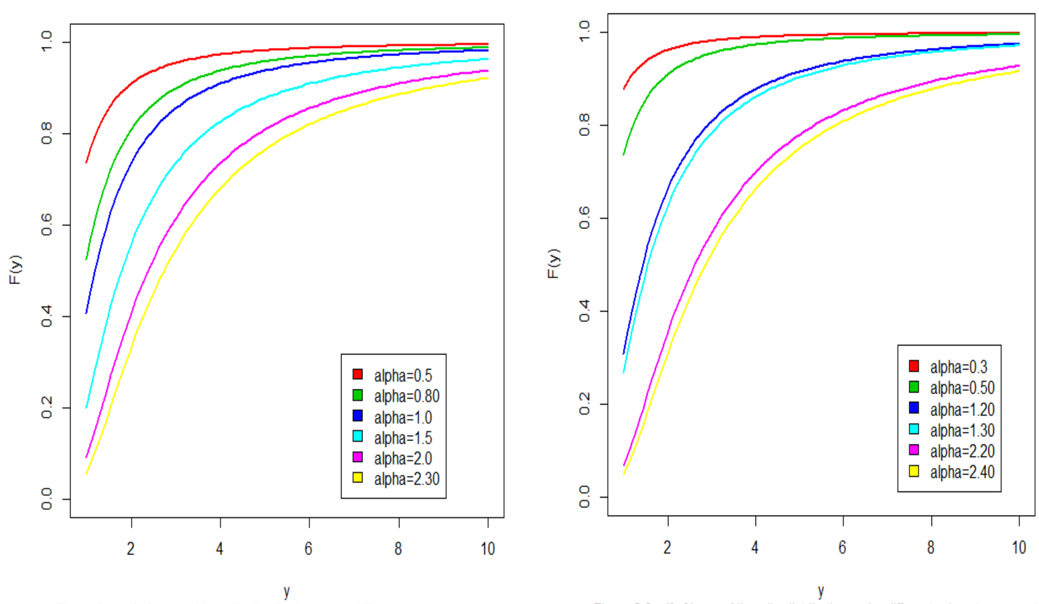


Figure 2.1:cdf of iverse Ailamujia distribution under different values to parameter

Figure 2.2:cdf of iverse Ailamujia distribution under different values to parameter

Figs. 2.1 and 2.2. Illustrates different shapes of c.d.f of inverse Ailamujia distribution

3 Structural Properties of Inverse Ailamujia Distribution (IAD)

3.1 Moments of inverse Ailamujia distribution

Let Y be a random variable follows inverse Ailamujia distribution. Then the r^{th} moment denoted by μ_r' of the distribution is obtained as

$$\mu_r' = E(Y^r) = \int_0^\infty y^r\, f(y,\alpha)\, dy$$

$$= \int_0^\infty y^r 4\alpha^2 \frac{1}{y^3} e^{-\frac{2\alpha}{y}}\, dy$$

$$= 4\alpha^2 \int_0^\infty y^{r-3}\, e^{-\frac{2\alpha}{y}}\, dy$$

Making the substitution $\frac{2\alpha}{y} = t$ and after solving the integral, we obtain.

$$\mu_r' = E(Y^r) = (2\alpha)^r \Gamma(2-\alpha) \tag{2.3}$$

By substituting $r = 1$, we obtain the mean of the distribution

Mean $= \mu_1' = 2\alpha$

We observe that equation (2.3), does not exists for $r \geq 2$.which implies that higher order moments does not exists.

3.2 Moment generating function of inverse Ailamujia distribution

Let Y be a random variable follows inverse Ailamujia distribution then the moment generating function denoted by $M_Y(t)$ is obtained as.

$$M_Y(t) = \int_0^\infty e^{ty}\, f(y,\alpha) dy$$

Using taylors theorem, we get

$$= \int_0^\infty \left\{1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \cdots\right\} f(y,\alpha) dy$$

$$= \int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!} y^r f(y,\alpha) dy$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} \int_0^\infty y^r f(y,\alpha) dy$$

$$= \sum_{r=0}^\infty \frac{t^r}{r!} (2\alpha)^r \Gamma(2-r)$$

3.3 Characteristics function of inverse Ailamujia distribution

Let Y be a random variable follows inverse Ailamujia distribution then the moment generating function denoted by $\phi_Y(t)$ is obtained as.

$$\phi_Y(t) = \int_0^\infty e^{ity}\, f(y,\alpha) dy$$

Using taylors theorem, we get

$$= \int_0^\infty \left\{1 + ity + \frac{(ity)^2}{2!} + \frac{(ity)^3}{3!} + \cdots \right\} f(y, \alpha) dy$$

$$= \int_0^\infty \sum_{r=0}^{\infty} \frac{(it)^r}{r!} y^r f(y, \alpha) dy$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^\infty y^r f(y, \alpha) dy$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} (2\alpha)^r \Gamma(2 - r)$$

3.4 Harmonic mean of inverse Ailamujia distribution

The harmonic mean (H) is given as.

$$\frac{1}{H} = E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f(y, \alpha)\, dy$$

$$= 4\alpha^2 \int_0^\infty \frac{1}{y^4} e^{-\frac{2\alpha}{y}}\, dy$$

Making the substitution $\frac{2\alpha}{y} = t$ and after solving the integral, we obtain

$$\frac{1}{H} = \frac{1}{\alpha}$$

3.5 Mode and median of inverse Ailamujia distribution

Taking the log function to the p.d.f of inverse Ailamujia distribution, we have

$$\log f(y, \alpha) = 2 \log 2\alpha - 3 \log y - \frac{2\alpha}{y} \tag{2.4}$$

Differentiate (2.4), w.r.t y, we get

$$\frac{\partial \log f(y, \alpha)}{\partial y} = -\frac{3}{y} + \frac{2\alpha}{y^2} \tag{2.5}$$

Substituting $\frac{\partial \log f(y,\alpha)}{\partial y} = 0$, we get

$$y = \frac{2\alpha}{3} \Rightarrow M_0 = y_0 = \frac{2\alpha}{3}$$

Using the empirical formula for median , we get

$$M_d = \frac{1}{3} M_0 + \frac{2}{3} \mu_1^{'} = \frac{14\alpha}{9}$$

4 Shanon Entropy of Inverse Ailamujia Distribution

The concept of information entropy was introduced by Shanon in 1948. The entropy can be interpreted as the average rate at which information is produced by a random source of data and is given by

$$H(y,\alpha) = -E[\log f(y,\alpha)]$$

$$= -E\left[\log\left(4\alpha^2 \frac{1}{y^3} e^{-\frac{2\alpha}{y}}\right)\right]$$

$$= -E\left[2\log 2\alpha - 3\log y - \frac{2\alpha}{y}\right]$$

$$= -2\log 2\alpha + 3E(\log y) + 2\alpha E\left(\frac{1}{y}\right) \tag{4.1}$$

Now

$$E(\log y) = \int_0^\infty \log y\, f(y,\alpha)dy$$

$$= 4\alpha^2 \int_0^\infty \log y \frac{1}{y^3} e^{-\frac{2\alpha}{y}} dy$$

Making the substitution $\frac{2\alpha}{y} = t$ and after solvig the integral, we get

$$E(\log y) = \log 2\alpha - \Gamma'(2) \tag{4.2}$$

Also

$$E\left(\frac{1}{y}\right) = 4\alpha^2 \int_0^\infty \frac{1}{y} f(y,\alpha)dy$$

$$= 4\alpha^2 \int_0^\infty \frac{1}{y^4} e^{-\frac{2\alpha}{y}} dy$$

Making the substitution $\frac{2\alpha}{y} = t$ and after manuplating integral, we get

$$E\left(\frac{1}{y}\right) = \frac{1}{\alpha} \tag{4.3}$$

Substituting equations (4.2), (4.3) in (4.1), we get

$$H(y,\alpha) = -2\log 2\alpha + 3\{\log 2\alpha - \Gamma'(2)\} + 2$$

5 Survival Measures

The reliability function of a random variable $y$is denoted as $R(y,\alpha)$, can be obtained as

$$R(y,\alpha) = 1 - F(y,\alpha)$$

Using (2.1) in the above equation, we get

$$R(y,\alpha) = 1 - \frac{(2\alpha + y)}{y} e^{-\frac{2\alpha}{y}} \tag{5.1}$$

The hazard rate function denoted as $h(y,\alpha)$of a random variable y can be obtained as

$$h(y,\alpha) = \frac{f(y,\alpha)}{R(y,\alpha)} \tag{5.2}$$

using equation (2.1) and (5.1) in (5.2), we get

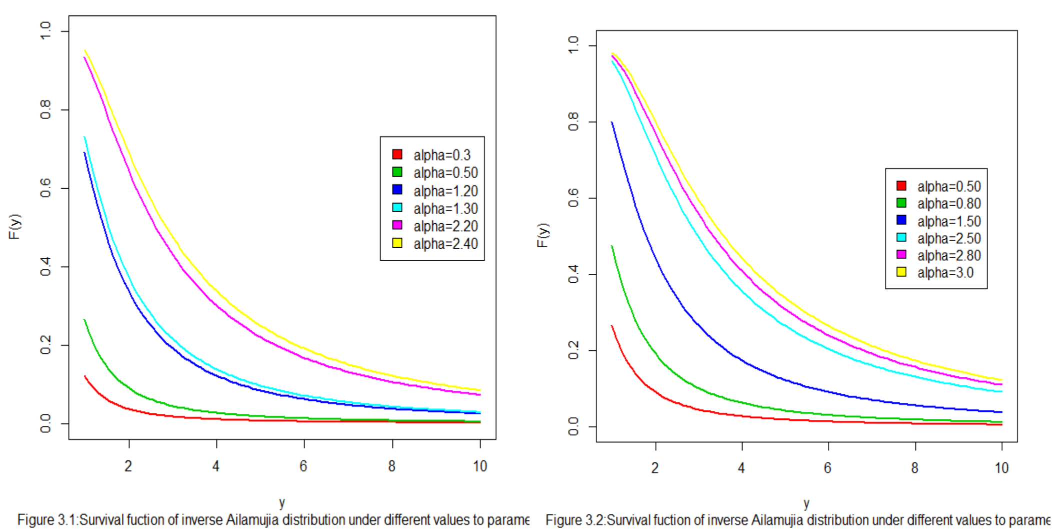
$$h(y,\alpha) = \frac{4\alpha^2 e^{-\frac{2\alpha}{y}}}{y^2\left\{y - (2\alpha + y)e^{-\frac{2\alpha}{y}}\right\}}$$

Also, the reverse hazard rate function denoted as $h_r(y,\alpha)$, can be obtained as

$$h_r(y,\alpha) = \frac{f(y,\alpha)}{F(y,\alpha)} \tag{5.3}$$

Using equation (2.1) and (2.2) in equation (5.3), we get

$$h_r(y,\alpha) = \frac{4\alpha^2}{y^2(2\alpha+y)}$$



Figs. 3.1 and 3.2. Illustrates different shapes of survival function of inverse Ailamujia distribution

6 Order Statistics of Inverse Ailamujia Distribution

Let us suppose $Y_1, Y_2, Y_3 \dots, Y_n$ be random samples of size n from inverse Ailamujia distribution with p.d.f $f(y)$ and c.d.f $F(y)$ .Then the probability density function of k^{th} order statistics is given as.

$$f_{Y(k)}(y) = \frac{n!}{(k-1)!\,(n-k)!}[F(y)]^{k-1}[1-F(y)]^{n-k}\,f(y), \quad k = 1,2,3,\ldots,n \tag{6.1}$$

Now substituting the equation (2.1) and (2.2) in equation (6.1) , we obtain the probability density function of k^{th} order statistics is given as

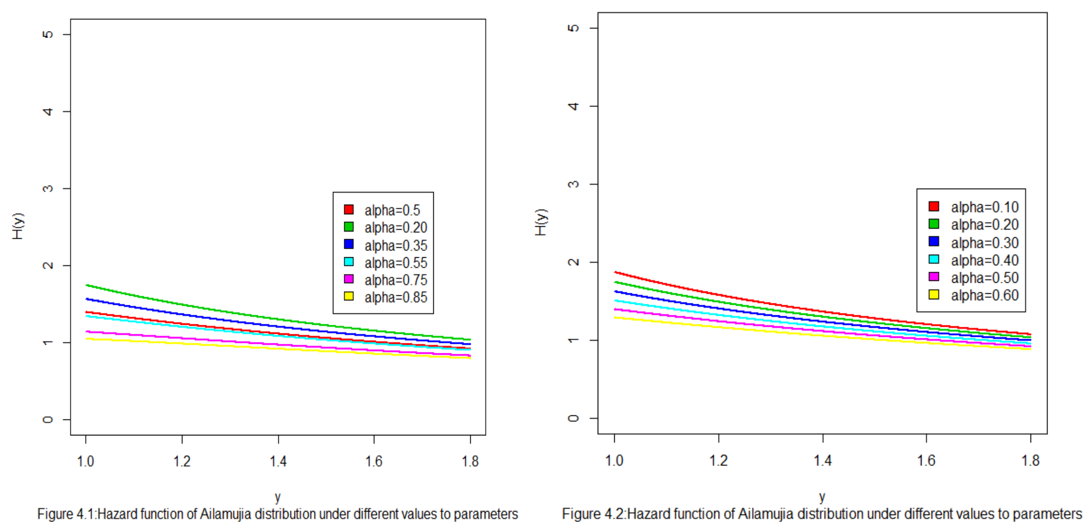
$$f_{Y(k)}(y) = \frac{n!\,4\alpha^2 e^{-\frac{2\alpha}{y}}}{(k-1)!\,(n-1)!}\left[(2\alpha+y)e^{-\frac{2\alpha}{y}}\right]^{k-1}\left[y-(2\alpha+y)e^{-\frac{2\alpha}{y}}\right]^{n-k} \tag{6.2}$$

The p.d.f of first order Y_1 is given as

$$f_{Y(1)}(y) = \frac{4n\alpha^2 e^{-\frac{2\alpha}{y}}}{y^{n+2}}\left[y-(2\alpha+y)e^{-\frac{2\alpha}{y}}\right]^{n-1}$$

And the n^{th} order probability density function Y_n is given as

$$f_{Y(1)}(y) = \frac{4n\alpha^2 e^{-\frac{2\alpha}{y}}}{y^{n+2}}\left[(2\alpha+y)e^{-\frac{2\alpha}{y}}\right]^{n-1}$$



Figs. 4.1 and 4.2. Illustrates different shapes of Hazard function of inverse Ailamujia

7 Estimation of Parameters Inverse Ailamujia Distribution

7.1 Methods of moments

The moment estimator for α denoted as $\hat{\alpha}$ can be obtained by equating population moments with sample moments given as

$$\mu_r' = \frac{1}{n}\sum_{i=1}^{n} y_i^r$$

Therefore we have

$$=\frac{1}{n}\sum_{i=1}^{n} y_i = 2\alpha$$

$$\bar{y} = 2\alpha \Rightarrow \hat{\alpha} = \frac{\bar{y}}{2}$$

7.2 Method of maximum likelihood estimation

Let $Y_1, Y_2 \dots Y_n$ be random samples from the inverse Ailamujia distribution. Then the likelihood function of inverse Ailamujia distribution is given as

$$l = \prod_{i=1}^{n} f(y_i, \alpha)$$

$$= \prod_{i=1}^{n} 4\alpha^2 \frac{1}{y_i^3} e^{-\frac{2\alpha}{y_i}} = (4\alpha^2)^n \prod_{i=1}^{n} \frac{1}{y_i^3} e^{-2\alpha \sum_{i=1}^{n} \frac{1}{y_i}}$$

Taking log we get log likelihood function as

$$\log l = 2n \log 2\alpha - 3\sum_{i=1}^{n} \log y_i - 2\alpha \sum_{i=1}^{n} \frac{1}{y_i}$$

Differentiating w.r.t α, we get

$$\frac{\partial \log l}{\partial \alpha} = 2n\frac{1}{2\alpha} - 2\sum_{i=1}^{n} \frac{1}{y_i}$$

Now equating $\frac{\partial \log l}{\partial \alpha} = 0$, we get

$$\hat{\alpha} = \frac{n}{2S}$$

Where $S = \sum_{i=1}^{n} y_i^{-1}$

8 Data Analysis

Data Set 1: In this section we provide an applications which explains the performance of the newly distribution. The data set has been taken from Gross and Clark [10], which signifies the relief times of 20 patients getting an analgesic. We use previous data to associate the fit of the newly developed model with Ailamujia, Exponential, Inverse Exponential, Lindley, Inverse Lindley distribution. The data are follows.

1.1,1.4,1.3,1.7,1.9,1.8,1.6,2.2,1.7,2.7,4.1,1.8,1.5,1.2,1.4,3.0,1.7,2.3,1.6,2.0.

In order to compare the two distribution models, we consider the criteria like AIC (Akaike information criterion, AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion. The better distribution corresponds to lesser AIC, AICC and BIC values.

$AIC = -2lnL + 2k, AICC = AIC + \frac{2k(k+1)}{(n-k-1)}, BIC = -2lnL + klnn$

Table 8.1. ML estimates and criteria for comparison

Distribution	Estimates	-2log*L*	AIC	AICC	BIC
Inverse Ailamujia Distribution	1.7247	51.6526	53.6526	53.9859	54.6483
Ailamujia Distribution	0.5263	52.3262	54.3262	54.6595	55.3219
Exponential Distribution	0.5263	65.674	67.674	68.0073	68.6697
Inverse Exponential Distribution	1.724	65.3372	67.3372	67.6705	68.3329
Lindley Distribution	0.8161	60.499	62.499	62.8323	63.499
Inverse Lindley Distribution	2.2546	63.5142	65.5142	65.8475	66.5099

Data Set 2: This data set is the strength data of glass of the aircraft window reported by Fuller et al. [11].

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.80, 26.69, 26.77, 26.78
27.05, 27.67, 29.90, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08,
37.09, 39.58, 44.045, 45.29, 45.381

Table 8.2. ML estimates and criteria for comparison

Distribution	Estimates	-2log*L*	AIC	AICC	BIC
Inverse Ailamujia Distribution	29.2153	252.2188	254.2188	254.4256	255.6527
Ailamujia Distribution	0.03245	252.2306	254.2306	254.4374	255.6645
Exponential Distribution	0.0324	274.53288	276.53288	276.73977	277.96678
Inverse Exponential	29.2152	274.523	276.523	276.72989	277.9569
Lindley Distribution	0.06298	253.9884	255.9884	256.1952	257.4223
Inverse Lindley Distribution	30.1531	700.416	702.416	702.6228	703.8499

From Table 8.1 and 8.2, it has been observed that the inverse Ailamujia distribution (AID) have the lesser AIC, AICC, -2logL and BIC values as compared to Ailamujia, Exponential, Inverse Exponential, Lindley, Inverse Lindley distribution distributions. Hence we can conclude that inverse Ailamujia distribution leads to a better fit as compared to Ailamujia, Exponential, Inverse Exponential, Lindley, Inverse Lindley distributions.

9 Conclusion

In this paper the inverse analogue of Ailamujia distribution has been established. Many times inverse of the distributions provide better results for fitting different data taken from various fields. Some statistical properties including moments, moment generating function, characteristics function, mode, median, shanon entropy, hazard rate function, reverse hazard function has been discussed. The estimation of the parameters of the established distribution has been estimated by the method of moments and maximum likelihood estimator. Finally two real life data sets have been presented to check the performance of the established model.

Competing Interests

Authors have declared that no competing interests exist.

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