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On Almost Semi-Invariant Submanifold of A Normal Almost Paracontact Manifold

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

In the present paper we have obtained some properties of an almost semi-invariant of a normal almost paracontact manifold. The integrability condition of distributions $D, D^{\perp}, D \oplus \{\xi\}$ have also been discussed. According to these cases normal almost paracontact manifold is categorized and its used to demonstrate that the method presented in this paper is effective.

Keywords: Almost Semi invariant submanifold; Normal almost paracontact manifold.

MSC (2010): 53C15 – General geometric structures on manifolds (almost complex, almost product structures, etc.).

1 Introduction

A Riemannian manifold $(\widetilde{M}, g)$ is called almost paracontact metric manifold if it is endomed with structure (ϕ, ξ, η, g), where ϕ is a (1, 1) tensor, ξ and η vector field and 1-form on $\widetilde{M}$, respectively, satisfying.

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$$\phi^2 X = X - \eta(X)\xi \quad (1.1)$$

$$\phi\xi = 0, \eta\, 0\, \phi = 0 \quad (1.2)$$

$$\eta(\xi) = 1 \quad (1.3)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (1.4)$$

$$\eta(X) = g(X, \xi) \quad (1.5)$$

for any $X, Y \in (TM)$, where TM denotes the set of all smooth vector fields on $\widetilde{M}$ [1]

An almost paracontact metric manifold $\widetilde{M}$ is said to be normal if the covariant derivative of ϕ satisfies

$$(\widetilde{\nabla}_X \phi)Y = -g(X, Y)\,\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \quad (1.6)$$

And

$$\widetilde{\nabla}_X \xi = \phi X \quad (1.7)$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$

Let $\widetilde{\nabla}$ (resp. ∇) be the linear connection of $\widetilde{M}$ (resp. M) with respect to the Riemannian metric g. The linear connection induced by $\widetilde{\nabla}$ on the normal bundle $TM^{\perp}$ is denoted by $\nabla^{\perp}$ then the equation of Gauss and Weingarten are respectively given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (1.8)$$

And

$$\widetilde{\nabla}_X N = -A_N X + \nabla_X{}^{\perp} N \quad (1.9)$$

For all $X \in \lceil(TM)$ and $N \in \lceil(TM^{\perp})$, h is the second fundamental form of M and A_N is the fundamental tensor with respect to the normal section N and

$$g(h(X, Y), N) = g(A_N X, Y) \quad (1.10)$$

Let M be an m-dimensional submanifold immersed of a normal almost paracontact manifold $\widetilde{M}$. Let $TM \; and \; T^{\perp}M$ be respectively the tangent and normal bundle to M. Suppose the structure vector field ξ is tangent to M and denoted by $\{\xi\}$ the one dimensional distribution spanned by ξ on M and $\{\xi\}^{\perp}$ the complementary orthogonal distribution to $\{\xi\}$ in TM. For each $X \in (TM)$, Put

$$\phi X = bX + cX \quad (1.11)$$

where $bX \in (\{\xi\}^{\perp}) \; and \; cX \; (T^{\perp}M)$. Thus b is an endomorphism of the tangent bundle TM and c is a normal bundle 1-form on M.

1.1 Definition

The submanifold of a normal almost para contact manifold is said to be an almost semi-invariant submanifold and its tangent bundle TM has the decomposition

$$TM = D \oplus D^{\perp} \oplus D\,\{\xi\} \quad (1.12)$$

where

(a) D is invariant distribution on M i.e. $\phi(D_x) = D_x$

(b) $D^{\perp}$is an anti-invariant distribution on M, i.e. $\phi D_x \subset (T^{\perp}M)$ for $X \in M$

(c) $\widetilde{D}$ is neither invariant nor an anti-invariant distribution on M, i.e. $bX_x \neq 0 \; and \; cX_x \neq 0$ for any $x \in M \; and \; X_x \in D_x$

(d) $\{\xi\}$ is the distribution spanned in M by the Vector field ξ

Para Contact manifolds and almost semi invariant submanifolds were studied by many investigators (See, [2 – 7, 8 – 11])

2 Basic Results

Let M be an almost semi-invariant submanifold of a normal almost paracontact manifold $\widetilde{M}$ and M both by g. Let P, Q, and L be the projection morphisms of TM to the distributions, $D, D^{\perp} \; and \; \widetilde{D}$, respectively. Then,$X \in (TM)$, we have

$$X = PX + QX + LX + \eta(X)\xi \tag{2.1}$$

Now, we take $X \in (\widetilde{D})$. Then $bX \neq 0 \, , cX \neq 0$. Thus c defines a vector sub-bundle $c\widetilde{D} : x \longrightarrow c\widetilde{D}_x$ of $TM^{\perp}$. For any $N \in (T^{\perp}M)$, we put

$$\phi N = tN + fN \tag{2.2}$$

where tN and fN are respectively the tangent and the normal component of ϕN. Then we have

$$g(\phi D^{\perp} , c\widetilde{D}) = 0 \tag{2.3}$$

Next, we denote by υ the orthogonal complementary vector bundle to $\phi D^{\perp} \oplus c\widetilde{D} \; in \; (T^{\perp}M)$. By (1.4), we have

$$g(\phi X, cy) = g(\phi X, \phi Y) = g(X, Y) = 0, for \; X \in (D) and \; Y \in (\widetilde{D}) \tag{2.4}$$

Thus $T^{\perp}M = \phi D^{\perp} \oplus C\widetilde{D}$ (2.5)

2.1 Lemma

The morphism t and f satisfy

$$t(TM^{\perp}) = D^{\perp} \oplus \widetilde{D} \tag{2.6}$$

$$t(\phi D^{\perp}) = D^{\perp} \tag{2.7}$$

$$t(c\widetilde{D}) = \widetilde{D} \tag{2.8}$$

$$f(c\widetilde{D}) = c\widetilde{D} \tag{2.9}$$

<u>Proof</u> Let $N \in (T^{\perp}M)$, then

$g(tN, X) = g(\phi N, X)$

$= g(N, \phi X) = 0 \quad \forall \quad X \in (D)$
And, $g(tN, \xi) = g(\phi N, \xi)$

$= g(N, \phi\xi) = 0$

Thus, $tN \in \left(D^{\perp} \oplus \widetilde{D}\right)$ and we get (2.6). Next for each $X \in (D^{\perp})$, we have

$X = \phi^2 X = t\phi X + c\phi X = t\phi X$

Which implies (2.7)

We now have

$g(tcX, Z) = g(\phi cX, Z),\ for\ Z \in (D^{\perp}), X \in \left(\widetilde{D}\right)$

$= g(cX, \phi Z) = 0$

and $g(tcX, \xi) = g(\phi cX, \xi) = g(cX, \phi\xi) = 0$

$g(tcX, Y) = g(\phi cX, Y) = g(cX, \phi Y) = 0, \quad \forall\, Y \in (D)\ and\ X \in \widetilde{D}$

Therefore,

$tcX \in \lceil\left(\widetilde{D}\right)$

Giving (2.8)

Lastly, we have

$g(fcX, N) = g(\phi cX, N)$, where $N \in \upsilon, X \in \left(\widetilde{D}\right)$

$= g(cX, \phi N) = 0$

And

$g(fcX, \phi z) = g(\phi cX, \phi Z)$ for $Z \in (D^{\perp})$ and $X \in \left(\widetilde{D}\right)$

$= g(cX, Z) = 0$

And hence we get (2.9)

2.2 Lemma

Let M be an almost semi-invariant submanifold of a normal almost para contact manifold $\widetilde{M}$. Then we have,

$$(b^2 + tc)X = X - \eta(X)\xi,\ (tc + fc)X = 0 \tag{2.10}$$

$$(f^2 + ct - I)N = 0, (bt + tf)N = 0 \tag{2.11}$$

$$(f^3 - f + ctf)N = 0 \tag{2.12}$$

$(b^3X - b + tcb)X = 0$ (2.13)

For any $X \in (TM)$ and $N \in (T^{\perp}M)$

Proof The proof follows directly from (1.1), (1.11) and (2.2)

Proposition 2.1 Let M be an almost semi-invariant submanifold of a normal almost paracontact manifold $\widetilde{M}$. Then the endomorphism $b: TM \longrightarrow TM$ is a para f-structure on M, that is, $b^3 - b = 0$ if and only if M is a semi-invariant submanifold.

Proof From (1.11), we see that

$(b^3 - b)X = 0$ for any $X \in \left(D \oplus \widetilde{D} \oplus \{\xi\}\right)$

Since $b\widetilde{D}_x = \widetilde{D}_x$, we see that

$(b^3 - b)\left(\widetilde{D}\right) = \{0\}\ if\ and\ only\ if$

$(b^2 - I)\left(\widetilde{D}\right) = 0$

Which with the help of (2.10) gives

$(b^2 - I) = -tc$

Therefore $tc\left(\widetilde{D}\right) = 0$

Which with the help of (2.8) gives $\widetilde{D} = \{0\}$.

Proposition 2.2 Let M be an almost semi-invariant sub-manifold of a normal almost paracontact manifold. Then M is a semi-invariant sub manifold if and only if $(f^3 - f) = 0$ (2.14)

Proof We see that $N \in (\phi D^{\perp})$, we have $fN = 0$ and for $N \in (\upsilon),\ fN = \phi N$

By (2.9), we see that t is an automorphism on $c\widetilde{D}$.

Hence $(f^3 - f)\left(c\widetilde{D}\right) = \{0\}$ if and only if

$(f^3 - I)\left(c\widetilde{D}\right) = \{0\}$ (2.15)

Using (2.11), we get

$ct\left(c\widetilde{D}\right) = 0$

Which with the help of (2.8) gives $\widetilde{D} = \{0\}$.

2.3 Lemma

Let M be an almost semi-invariant submanifold of a normal almost paracontact manifold, then we have

$P\left(u(X,Y)\right) = \phi P \nabla_X Y - \eta(Y)PX$ (2.15)

$Q\left(u(X,Y)\right) = \phi Q \nabla_X Y - \eta(Y)QX$ (2.16)

$$L\big(u(X,Y)\big) = \phi L\nabla_X Y - \eta(Y)LX \tag{2.17}$$

$$-g(X,Y) + \eta(X)\eta(Y) = \eta\big(u(X,Y)\big) \tag{2.18}$$

$$\phi Q\nabla_X Y + bL\nabla_X Y + cL\nabla_X Y = h(X,\phi PY) + h(X,bLY) + \nabla_X{}^{\perp}\phi QY + \nabla_X cLY + fh(X,Y) \tag{2.19}$$

Where,

$$u(X,Y) = \nabla_X \phi PY + \nabla_X bLY - A_{\phi QY}X - A_{CLY}X \tag{2.20}$$

For all $X, Y \in (TM)$

Proof From (2.1), we see that

$$Y = PY + QY + L(X) + \eta(Y)\xi \tag{2.21}$$

Differentiating (2.21) covariantly along X and using (1.6), (2.1), (1.8) and (1.9), we get

$$\phi P\nabla_X Y + \phi Q\nabla_X Y + bL\nabla_X Y + cL\nabla_X Y + P\,th(X,Y) + Q\,th\,(X,Y) + L\,th(X,Y) + \eta(th(X,Y)\xi) + fh(X,Y) - g(X,Y)\xi + \eta(X)\eta(Y)\xi - \eta(Y)PX - \eta(Y)QX - \eta(Y)LX$$
$$= P\nabla_X\phi PY + Q\nabla_X\phi PY + L\nabla_X\phi PY + \eta(\nabla_X\phi PY)\xi + h(X,\phi PY) - PA_{\phi QY}X - QA_{\phi QY}X - LA_{\phi QY}X - \eta\big(A_{\phi QY}X\big)\xi + \nabla_X{}^{\perp}\phi QY + P\nabla_X bLY + Q\nabla_X bLY + L\nabla_X bLY + \eta(\nabla_X bLY)\xi + h(X,bLY) - PA_{cLY}X - QA_{cLY}X - L\,A_{cLY}X - \eta(A_{cLY}X)\xi + \nabla_X{}^{\perp}cLY$$

Equating tangent and normal parts, we get (2.15), (2.16), (2.17), (2.18) and (2.19)

2.4 Lemma

Let M be an almost semi-invariant submanifold of a normal almost paracontact manifold $\widetilde{M}$. Then we have

$$\nabla_X\,\xi = bX, h(X,\xi) = cX \quad \forall\, X \in (TM) \tag{2.22}$$

$$\nabla_X\,\xi = \phi X,\; h(X,\xi) = 0 \quad for\ any\ X \in (D) \tag{2.23}$$

$$\nabla_Y \xi = 0, h(Y,\xi) = \phi Y \quad for\ any\ Y \in (D^{\perp}) \tag{2.24}$$

$$\nabla_Z\,\xi = bZ, h(Z,\xi) = cZ\ for\ any\ Z \in \big(\widetilde{D}\big) \tag{2.25}$$

$$\nabla_\xi\,\xi = 0, h(\xi,\xi) = 0 \tag{2.26}$$

Proof We have

$$\widetilde{\nabla}_X\xi = \nabla_X\xi + h(X,\xi)$$

Which on using (1.7) and (1.11), gives

$$bX + cX = \nabla_X\,\xi + h(X,\xi)$$

Equating the tangent and the normal parts we get (2.22) and (2.23) – (2.26) are obtained directly from (2.22)

2.5 Lemma

Let M be an almost semi-invariant submanifold of a normal almost paracontact manifold $\widetilde{M}$. Then we have

$A_{\phi X}Y + A_{\phi Y}X = 0$ for all $X, Y \in (D^{\perp})$ (2.27)

Proof With the help of (1.6), (1.8) and (1.10), gives

$$\begin{aligned} g(A_{\phi X}Y, Z) &= g(h(Y,Z), \phi X) \\ &= g(\tilde{\nabla}_Z Y, \phi X) \\ &= g(\phi \tilde{\nabla}_Z Y, X) \\ &= g(\tilde{\nabla}_Z \phi Y, X) - g\left((\tilde{\nabla}_Z \phi)Y, X\right) \\ &= -g(\phi Y, \tilde{\nabla}_Z X) = -g(A_{\phi Y}X, Z) \end{aligned}$$

for all $X, Y \in (D^{\perp})$ and $Z \in (TM)$ which implies (2.27)

2.6 Lemma

Let M be an almost semi-invariant submanifold of a normal almost paracontact manifold $\tilde{M}$. Then we have

$\nabla_{\xi} U \in (D)$ *for any* $U \in (D)$ (2.28)

$\nabla_{\xi} V \in (D^{\perp})$ *for any* $V \in (D^{\perp})$ (2.29)

$\nabla_{\xi} W \in (\tilde{D})$ *for any* $W \in (\tilde{D})$ (2.30)

The proof follows from Bejancu and Papaglumic (1984a, b).

Corollary (2.1)

Let M be an almost semi-invariant submanifold of a normal almost paracontact manifold $\tilde{M}$. Then we have

$[X, \xi] \in (D)$ *for any* $X \in (D)$ (2.31)

$[Y, \xi] \in (D^{\perp})$ *for any* $Y \in (D^{\perp})$ (2.32)

$[Z, \xi] \in (\tilde{D})$ *for any* $Z \in (\tilde{D})$ (2.33)

The proof immediately follows from Lemmas (2.4) and (2.6)

3 Integrability of Distributions

3.1 Theorem

Let M be an almost semi-invariant submanifold of a normal almost paracontact manifold $\tilde{M}$. Then the distribution D is integrable if and only if

$h(X, \phi Y) = h(Y, \phi X)$ (3.1)

Proof By using (2.23), we have

$$\begin{aligned} g([X,Y], \xi) &= g(\nabla_X Y - \nabla_Y X, \xi) \\ &= -g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \\ &= -g(\phi X, Y) + g(\phi Y, X) \end{aligned}$$

$= 0 \quad$ for all $X, Y \in (D)$

Next from (2.19), we have

$$h(X, \phi Y) = \phi Q \nabla_X Y + cL\nabla_X Y + f\big(h(X, Y)\big) \tag{3.2}$$

for any $X, Y \in (D)$

Hence, we have

$$h(X, \phi Y) - h(Y, \phi X) = \phi Q[X, Y] + cL([X, Y])$$

Which proves the theorem.

3.2 Theorem

The distribution $D^{\perp}$ is not necessarily integrable

Proof For $X, Y \in (D^{\perp})$, (2.20) gives

$$u(X, Y) = -A_{\phi Y} X$$

Applying ϕ to (2.15) and using (1.1), we get

$P\nabla_X Y = -\phi P\big(A_{\phi Y} X\big)$, for any $X, Y \in (D^{\perp})$

Which with the help of Lemma (2.5), gives

$$\begin{aligned} P([X, Y]) &= \phi P\big(-A_{\phi Y} X + A_{\phi X} Y\big) \\ &= -2\phi P\big(A_{\phi Y} X\big) \end{aligned}$$

Showing the non integrability of $D^{\perp}$

3.3 Theorem

The distribution $\widetilde{D}$ is integrable if and only if

$$A_{CX} Y - A_{CY} X + \nabla_X bY - \nabla_Y bX \in \big(D^{\perp} \oplus \widetilde{D} \oplus \{\xi\}\big) \tag{3.3}$$

$$h(bX, Y) - h(X, bY) + \nabla_Y{}^{\perp} cX - \nabla_X{}^{\perp} cY \in (c\widetilde{D} + v) \tag{3.4}$$

for all $X, Y \in (\widetilde{D})$

Proof

For any $X, Y \in \big(\widetilde{D}\big)$, using (2.25), we get

$$\begin{aligned} g([X, Y]) &= g(\nabla_X Y - \nabla_Y X, \xi) \\ &= g(X, \nabla_X \xi) - g(Y, \nabla_X \xi) \\ &= g(X, bY) - g(Y, bX) \\ &= 0 \end{aligned}$$

Now, for any $X, Y \in (\widetilde{D})$, (2.20), gives

$$u(X,Y) = \nabla_X bY - A_{CY}X \tag{3.5}$$

Applying ϕ to (2.15) and using (1.1), we get

$$P\nabla_X Y = \phi P(\nabla_X bY - A_{CY}X)$$

And hence

$$P([X,Y]) = \phi P(\nabla_X bY - A_{CY}X - \nabla_Y bX + A_{CX}Y)$$

Which shows that $[X,Y] \in (\widetilde{D})$ if and only if (3.3) is satisfied. Further applying ϕ to (2.19), and taking the component in $D^{\perp}$, we get

$$Q\nabla_X Y = Qt\big(h(X,bY)\big) + \nabla_X{}^{\perp}cY - f\big(h(X,Y)\big)$$

Which further yields

$$Q([X,Y]) = Qt\big(h(X,bY)\big) + \nabla_X{}^{\perp}cY - \nabla_Y^{\perp}cX$$

Hence, $\widetilde{D}$ is integrable if and only if (3.4) is satisfied.

4 Discussion and Analysis of Result

In this paper I have given some results on almost semi-invariant submanifold and some properties on almost semi-invariant submanifold of a normal almost para contact manifold have been discussed.

5 Conclusion/Remarks

In this paper I have categorized normal almost paracontact manifold satisfying the conditions, $(\widetilde{\nabla}_X\phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \; and \; \widetilde{\nabla}_X\xi = \phi X$. This derivative operators are very important. It provides information about the structure on the manifold.

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Competing Interests

Author has declared that no competing interests exist.

References

[1] Sato I. On a Structure similar to the almost contact structure I and II, Tensor, N.S., 30 (1976), 219 – 224 and tensor, N.S. 1977;31:199–205.

[2] Bejancu A, Papaghuic N. Almost semi-invariant submanifold of a Sasakian manifold, Bull. Math. De la Soc. Dela R.S de Roumanic. 1984a;2(76):321-338.

[3] Bejancu A, Papaghuic N. Semi-invariant sub manifolds of a Sasakian space form, collog. Math. 1948b;48:77-88.

[4] Chen BY, CR-Submanifolds of a Kaehler manifold I, J. Differential Geometry. 1981;16:305-322:493–509.

[5] Bagewasi CS, Siddesha MS. Semi-invariant submanifold of $(LCS)_n$ – manifold, Commun. Korean Math. Soc. 2017;1–9.

[6] Bagewadi CS. Nirmala D, Siddesha MS. Semi-invariant Submanifolds of (K, u) contact manifold, Bull. Cal. Math. Soc. 2017;109(2):93-100.

[7] Pandey HB, kumar A. Anti-invariant submanifolds of almost para contact manifolds Indian J. Pure Apple. Math. 1985;16:586-590.

[8] Yano K, Kon M. On Contact CR-Submanifold, J. Korean Math. Soc. 1989;26:231-262.
– CR-Submanifolds of Kaehlerian and Sasakian manifolds, Birkhouser, Boston. 1983;361–364.

[9] Kupli Erken. On normal almost paracontact metric manifold of fimensions. Facta Universitatis (nis), ser. Math. Inform. 2015;5:777–788.

[10] Mehmet Atceken, Siraj Uddin. Semi-invariant submanifold of a normal almost paracontact manifold, Filomat. 2017;15:4875-4887.
Available:https://doi.org/10.2298/FIL1715875A. Published by Faculty of Sciences and Math., University of Nis, Serbia.

[11] Zamkovoy S. Cannonical Connections on paracontact manifolds, ann. Global Anal. Geom. 2009;36:37–60.

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