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Common Fixed Point Theorems for Commuting Self Maps in Banach Spaces via a Measure of Non-compactness

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

This paper provides common fixed point theorems for a commuting family of self mappings in Banach spaces via a measure of non-compactness. The choice of a commuting family of self maps provides a result that is unique in its own right, generalize Darbos fixed point theorem and specifically extend and improve the work of Meryeme El Harrak and Ahmed Hajji.

Keywords: Fixed point; common fixed point; commuting family; self maps; measure of non-compactness; banach space; compact set.

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1 INTRODUCTION

Schauders fixed point theorem has been acclaimed as one of the well known theorems in fixed point theory. As a generalization of the above results, several other researchers have contributed immensely to extend Schauders results. Among these great minds was Darbo [1], who generalized Schauders theorem via a contraction condition in terms of a measure of noncompactness.

There have been many extensions of Darbos fixed point theorem. Recently, Hajji-Hanebaly [2]

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proved some fixed point theorems for a pair of commuting operators which generalized Darbos. Again, in a recent paper Hajji [3] generalized the theorems of Darbo [1], Sadovski and Markov-Kakutani.

In [4], Khodabakhshi and Vaezpour obtained new fixed point results using a technique associated with a measure of non-compactness for two commuting operators. Finally, Meryeme El Harrak and Ahmed Hajji established in a recent paper some new contraction condition which gave rise to common fixed point theorems for two and three mappings. The obtained result of Meryeme-Hajji [5] was a specific generalization of Darbos.

In this present paper, my aim is to make use of some properties of a measure of non-compactness to establish contraction conditions for a family of commuting self maps in Banach spaces. The motivation for the concept of a commuting family of contractions stems from the work of Frimpong and Prempeh [6]. They obtained fixed point results in 2017 in reflexive Banach space using a family of maps.

2 PRELIMINARIES

This section begins with some notations, definition and ancillary facts which will be needed in subsequent developments.

2.1 The Concept of Non-compactness

The notion of a measure of non-compactness was first introduced by Kuratowski [7]. This notion emerged from the fact that every bounded set can be covered by a single ball of some radius. Again, several balls of smaller radii can also cover a bounded set. Since a compact set is totally bounded, it can be covered by finitely many balls of arbitrary small radii. To this end, the ball measure of non-compactness is defined below.

Definition 2.1

Let M be a metric space and E a nonempty subset of M. If B_E is the collection of all bounded subsets of E then the mapping $m : B_E \to [0, \infty)$ defined by

$$m(A) = \inf\{r > 0 \text{ : there exist finitely many balls of radius } r \text{ which cover} A\}$$

is called the ball measure of non-compactness of the set A.

Definition 2.2 [7]

Let M be a metric space, E a nonempty subset of M and B_E the family of all bounded subsets of E. A mapping $\mu : B_E \to [0, \infty)$ defined by
$\mu(H) = \inf\{d > 0 :$ there exist finitely many sets of diameter at most d which cover $H\}$ is called the Kuratowski measure of non-compactness of H.

Now, for any ball of radius $r > 0$, its diameter $d \leq 2r$. Thus, we have the relation

$$m(A) \leq \mu(A) \leq 2m(A),$$

for any nonempty bounded subset A of E.

Now, by virtue of the fact that both measures in Definitions **2.1** and **2.2** share many properties in common, we denote either of them by φ in the sequel. Thus, we have Definition **2.3** below.

Definition 2.3

The mapping $\varphi : B_E \to [0, \infty)$ has the following properties:
$K_1 :$ $\varphi(A) < \infty$ for any nonempty bounded subset A of E.

$K_2:\ \varphi(A)=\varphi(\bar{A})$ for any nonempty bounded subset A of E.
$K_3:\ A\subseteq B\Rightarrow\varphi(A)\leq\varphi(B)$
$K_4:\ \varphi$\{conv(A)\} $=\varphi(A)$ for any bounded subset A of E, where conv(A) is the convex hull of A.
$K_5:$ If A is compact then $\varphi(A)=0$. Conversely, if $\varphi(A)=0$ and A is complete then A is compact.
$K_6:\ \varphi(A\cup B)=\max\{\varphi(A),\varphi(B)\}$ for any pair of nonempty bounded subsets A,B of E.
$K_7:\ \varphi\{\alpha A+(1-\alpha)B\}\leq\alpha\{\varphi(A)\}+(1-\alpha)\{\varphi(B)\}$, for any $\alpha\in[0,1]$.
$K_8:$ The family $\ker\varphi=\{A\in B_E:\varphi(A)=0\}$ is a nonempty set and it is called the kernel of the measure of non-compactness. Thus $\ker\varphi$ consists of all nonempty compact subsets of E.
$K_9:$ If $\{A_n\}$ is a nested sequence of closed sets from B_E and $\lim\limits_{n\to\infty}\varphi(A_n)=0$, then the intersection set

$$A_\infty=\cap_{n=1}^{\infty}A_n$$

is nonempty, and since

$$\varphi\{A_\infty\}=\varphi\{\cap_{n=1}^{\infty}A_n\}\leq\varphi\{A_n\}$$

for any n, we have $\varphi\{\cap_{n=1}^{\infty}A_n\}=0$. Therefore $\varphi\{A_\infty\}=0$ and $A_\infty\in\ker\varphi$.

Definition 2.4

Let H be a nonempty, bounded, closed and convex subset of a Banach space E and let 2^H denote the power set of H. A mapping $\psi:2^H\to[0,1)$ is said to be affine and relatively measure of non-compactness if it obeys the following conditions:
$P_1:\ \psi\{\alpha x+(1-\alpha)y\}=\alpha\{\psi(x)\}+(1-\alpha)\{\psi(y)\}$, whenever $\alpha\in(0,1)$, and $x,y\in A$ for any $A\in 2^H$.
$P_2:$ The family $\ker\ \psi=\{A\in 2^H:\psi(A)=0\}$ is a nonempty set.
$P_3:\ A\subseteq B\Rightarrow\psi(A)\leq\psi(B)$.
$P_4:$ If $\{A_n\}$ is a sequence of closed sets such that $A_{n+1}\subseteq A_n,\ n\geq 1$ and $\lim\limits_{n\to\infty}\psi(A_n)=0$ then the intersection set $A_\infty=\cap_{n=1}^{\infty}A_n$ is nonempty.

It is worthy of note to recall three outstanding theorems which have separately played key roles in their own right in fixed point theory.

Theorem 2.1 (Schauder)

Let K be a nonempty, compact and convex subset of a Banach space E. Then each continuous map $T:K\to K$ has at least one fixed point in K.

A generalization of Schauders theorem known as Darbos fixed theorem is stated below.

Theorem 2.2 [1]

Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T:\Omega\to\Omega$ be a continuous mapping. Assume that there exists a constant $\alpha\in(0,1)$ such that

$$\varphi(TA)\leq\alpha\varphi(A),\ \text{for any subset } A \text{ of } \Omega.$$

Then T has a fixed point in Ω.

Next the theorem due to Caristi is presented below.

Theorem 2.3 [9]

Let (M, ρ) be a metric space and let $\phi : M \to \mathbb{R}$ be a lower semi-continuous function which is bounded from below. Suppose $T : M \to M$ is an arbitrary mapping such that

$$\rho(\zeta, T\zeta) \leq \phi(\zeta) - \phi(T\zeta)$$

for any $\zeta \in M$. Then T has a fixed point in M.

In 2013 Aghajani et al [10] proved a theorem which was remarkable in its own right. This result by Aghajani et al is among the theorems playing key roles in fixed point theory today.

Theorem 2.4 [10]

Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that

$$\psi\{\mu(TX)\} \leq \psi\{\mu(X)\} - \varphi\{\mu(X)\}$$

for any nonempty subset X of Ω, where μ is an arbitrary measure of non-compactness, and

$$\psi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \varphi : \mathbb{R}_+ \to \mathbb{R}_+$$

are given functions such that ψ is continuous on $\mathbb{R}_+$ and φ is lower semi-continuous such that $\varphi(0) = 0$ and $\varphi(t) > 0$ for any $t > 0$. Then T has at least a fixed point in Ω.

It is worth noting that each of the theorems stated above dealt with a single continuous map. A generalization of Darbos fixed point theorem with two continuous commuting mappings was thus provided by Hajji [3] in the theorem below.

Theorem 2.5 [3]

Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let T and S be two continuous mappings from Ω into Ω such that;
H_1 : $TS = ST$.
H_2 : T is affine
H_3 : There exists a constant $k \in (0, 1)$ such that for any $A \subseteq \Omega$, we have $\mu\{TS(A)\} \leq k\mu(A)$. Then the set $\{x \in \Omega : Tx = Sx = x\}$ is nonempty and compact.

Finally El Harrak and Ahmed Hajji provided what has been the most outstanding and remarkable generalization of Darbos theorem. They obtained a common fixed point theorem for three commuting mappings in Banach spaces.

Theorem 2.6 [5]

Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let T, S and H be three continuous mappings from Ω into Ω such that the following three conditions hold:
A_1 : H and S are affine.
A_2 : $TS = ST,\ TH = HT,\ SH = HS$.
A_3 : For any nonempty subset A of Ω, we have $\sigma(HA) \leq \varphi(SA) - \varphi\{S(\overline{\text{conv}}(TA))\}$, where
$\varphi : P(\Omega) \to [0, \infty)$ and $\sigma : P(\Omega) \to [0, \infty)$ are mappings such that σ satisfies conditions K_2, K_3, K_4 and K_8 in Definition **2.3**. Then each of T, S and H has a fixed point in Ω and T, S and H have a common fixed point in Ω.

3 MAIN RESULTS

This section is devoted to proving the main results of this paper. Under this, the main results on common fixed point theorems for a commuting family of self mappings will be proved. To this end some auxiliary facts which will be needed in the sequel are given below.

Theorem 3.1 [6]

Let Ω be a Banach space and let Λ be a bounded, closed and convex subset of Ω . Let $\{T_n\}, n \in \mathbb{N}$ and $n \geq 1$, be a sequence of contractions on Λ such that $T_n(x) \leq T_{n+1}(x),\ \forall\ n \geq 1, x \in \Omega$. If T_n converges point-wise on Λ to a contraction T then the convergence is uniform.

Lemma 3.1

A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be continuous at a point $x_0 \in \mathbb{R}_+$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\psi(B(x_0, \delta)) \subset\ B(Tx_0, \varepsilon)$. We say that ψ is continuous on $\mathbb{R}_+$ if it is continuous at all points $x \in \mathbb{R}_+$.

Lemma 3.2 [8]

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a lower semi-continuous function such that $\phi(0) = 0$ and $\phi(\tau) > 0$ for any $\tau > 0$. Then ϕ is a non-decreasing function with $\lim\limits_{n\to\infty} \phi^n = 0$.

Theorem 3.2

Let Ω be Banach space and let Λ be a bounded, closed and convex subset of Ω. Let
$T_k :\ \Lambda \to \Lambda,\ k \geq 1$ be a sequence of contractions on Λ which satisfy the following conditions:
$F_1 : T_jT_k = T_kT_j$ whenever $j \neq k$.
$F_2 : T_k,\ k = 1, 2, 3, \cdots n - 1$ are affine.
F_3 : There exists $\alpha \in [0, 1)$ such that for any $A \subseteq \Lambda$, we have $\varphi\{T(A)\} \leq \alpha\varphi(A)$ where φ is the measure in Definition **2.3** and $T = T_1T_2T_3 \cdots T_n$. Then the set $\{\tau \in \Lambda : T_j(\tau) = T_k(\tau) = \tau\}$ for any pair T_j, T_k is nonempty and compact.

Proof.
Let $\{\Lambda_n\}$ be a sequence defined as

$$\Lambda_0 = \Lambda \text{ and } \Lambda_n = \overline{\text{conv}}\{T(\Lambda_{n-1})\} \text{ for } n = 1, 2, 3, \cdots .$$

For $n = 1$,

$$\Lambda_1 = \overline{\text{conv}}\{T(\Lambda_0)\} \text{ which gives } \Lambda_1 \subseteq \Lambda_0$$

Next, assume that $\Lambda_n \subseteq \Lambda_{n-1}$ for some $n \geq 1$.

Then $\Lambda_{n+1} = \overline{\text{conv}}\{T(\Lambda_n)\} \subseteq \overline{\text{conv}}\{T(\Lambda_{n-1})\} = \Lambda_n$.

Thus by induction $\Lambda_n \subseteq \Lambda_{n-1}\ \forall\ n \geq 1$, showing that Λ_n is a nested sequence of compact sets.

Now consider the operator

$$T(\zeta) = \alpha T_k(\zeta) + (1 - \alpha)T_k(\zeta),$$

where $T = T_1T_2T_3\cdots T_n$ and $k = 1, 2, 3, \cdots (n-1)$.

Clearly the operator T maps Λ into itself, commutes with each T_k for $k = 1, 2, 3, \cdots (n-1)$, and is continuous.

Now, for any $A \subseteq \Lambda$,
$\varphi\{T(A)\} = \varphi\{\alpha T_k(A) + (1-\alpha)T_k(A)\} \leq \varphi\{\alpha T_k(A)\} + \varphi\{(1-\alpha)T_k(A)\} = \alpha\varphi\{T_k(A)\} + (1-\alpha)\varphi\{T_k(A)\} = \alpha^2\varphi(A) + (1-\alpha)\varphi T_k(A)$.

Therefore,

$$\varphi\{T(A)\} \leq \alpha^2\varphi(A) + (1-\alpha)\{\varphi T_k(A)\}.$$

Since $\alpha \in (0,1),\ \alpha^2 < \alpha$. Hence $\alpha^2 + 1 - \alpha < 1$. Therefore by Theorem **2.5** we conclude that the set $\{\tau \in \Lambda : T_j(\tau) = T_k(\tau) = \tau\}$ is nonempty and compact. □

Theorem 3.3

Let Ω be Banach space and let Λ be a bounded, closed and convex subset of Ω. Let
$T_k:\ \Lambda \to \Lambda,\ k \geq 1,$ be a sequence of contractions on Λ that satisfy the following conditions:
$F_1 : T_jT_k = T_kT_j$ whenever $j \neq k$.
F_2 : For any pair T_j, T_k and for any nonempty subset A of Λ we have

$$\sigma(A) \leq \psi\{T_k(A)\} - \psi\left\{T_k\left(\overline{\text{conv}}\big(T_j(A)\big)\right)\right\},$$

where σ and ψ are relatively measures of non-compactness. Then

1. T_k has at least a fixed point in Λ, for $k = 1, 2, 3, \cdots, n$
2. If at least one of the maps $T_k,\ k = 1, 2, 3, \cdots, (n-1)$ is affine then $T_k,\ k = 1, 2, 3, \cdots, n$ have a common fixed point in Λ.

Proof.

1. Let the sequence $\{\Lambda_n\}$ be constructed as $\Lambda_0 = \Lambda$ and $\Lambda_{n+1} = \overline{\text{conv}}\{T(\Lambda_n)\}$ for $n \geq 0$. Then

$$T\Lambda_0 = T\Lambda \subseteq \Lambda = \Lambda_0 \tag{3.1}$$
$$\Lambda_1 = \overline{\text{conv}}\{T(\Lambda_0)\} \subseteq \Lambda_0 \tag{3.2}$$

and again

$$\Lambda_2 = \overline{\text{conv}}\{T(\Lambda_1)\} \subseteq \Lambda_1. \tag{3.3}$$

Thus we obtain the nested sequence $\Lambda_0 \supseteq \Lambda_1 \supseteq \Lambda_2 \supseteq \cdots \supseteq \Lambda_n \supseteq \Lambda_{n+1} \supseteq \cdots$ in Λ which when compressed gives $\Lambda_n \subseteq \Lambda_{n-1}\ \forall\, n \geq 1$.
Now, for any pair T_j, T_k we have

$$\sigma(\Lambda_n) \leq \psi\{T_k(\Lambda_n)\} - \psi\big\{T_k\big(\overline{\text{conv}}\big(T_j(\Lambda_n)\big)\big)\big\} \tag{3.4}$$

$\implies\ \psi\{T_k(\Lambda_n)\} - \psi\left\{T_k\left(\overline{\text{conv}}\big(T_j(\Lambda_n)\big)\right)\right\} \geq 0\ \forall\, n \in \{0, 1, 2, \cdots\}$ since $\sigma(\Lambda_n) \geq 0$.
Thus

$$\psi\{T_k(\Lambda_{n+1})\} = \psi\left\{T_k\left(\overline{\text{conv}}\big(T_j(\Lambda_n)\big)\right)\right\} \tag{3.5}$$

By virtue of the fact that

$T_j(\Lambda_n) \subseteq \Lambda_n$ for any n and j, and thus

$$\left(\overline{\text{conv}}(T_j(\Lambda_n))\right) \subseteq \Lambda_n,$$

we get

$$\psi\left\{T_k\left(\overline{\text{conv}}(T_j(\Lambda_n))\right)\right\} \leq \psi\{T_k(\Lambda_n)\}.$$

Therefore,

$$\psi\{T_k(\Lambda_{n+1})\} \leq \psi\{T_k(\Lambda_n)\},\ \forall\, n \in \{1, 2, 3, \cdots\}$$

This means that $\psi\{T_k(\Lambda_n)\}$ is a nonnegative and non-increasing sequence in $\mathbb{R}$ and thus converges to an element $\tau \in \mathbb{R}_+$ as $n \to \infty$.
Using (3.4) and (3.5) we obtain

$$\sigma(\Lambda_n) \leq \psi\{T_k(\Lambda_n)\} - \psi\{T_k(\Lambda_{n+1})\}$$

$$\Longrightarrow \quad \lim_{n\to\infty} \sigma(\Lambda_n) = 0 \text{ by uniqueness of limit.}$$

Since the sequence $\{\Lambda_n\}$ is nested and $\Lambda_\infty = \cap_{n=1}^{\infty}\Lambda_n$ is nonempty, closed and convex and by reason of $\Lambda_\infty \subset \Lambda$, we see clearly that the set Λ_∞ is invariant under T_k for $k = 1, 2, 3, \cdots n$. Moreover $\Lambda_\infty \in \ker\sigma$. Thus by Theorem **2.1** T_k has at least a fixed point in Λ.

2. For a specific k, the set $F_{T_k} = \{x \in \Lambda : T_k(x) = x\}$ is a bounded set, closed and convex. Again $T_k(F_{T_k}) \subseteq F_{T_k}$ and by virtue of the fact that the operators $T_k,\ k = 1, 2, \cdots, n$ commute, we get $T_n(T_1T_2T_3\cdots T_{n-1})(x) = (T_1T_2T_3\cdots T_{n-1})T_n(x) = (T_1T_2T_3\cdots T_{n-1})(x)$. Thus for any $x \in F_{T_k},\ T_k(Tx) = Tx$, where $T = T_1T_2T_3\cdots T_{n-1}$. Therefore $T(F_{T_k}) \subset F_{T_k}$. The conclusion is that for any $A \subseteq F_{T_k}$,

$$\sigma(A) \leq \psi\{T_k(A)\} - \psi\left\{T_k\left(\overline{\text{conv}}(T(A))\right)\right\}$$

and T has a fixed point in F_{T_k}. Thus by extension each $T_k, k = 1, 2, 3, \cdots n$ has a fixed point in Λ.

□

Theorem 3.4

Let Ω be Banach space and let Λ be a bounded, closed and convex subset of Ω. Let
$T_k :\ \Lambda \to \Lambda,\ k \geq 1$ be a sequence of contractions on Λ that satisfy the following conditions:
$C_1 : T_k$ for $k = 1, 2, 3, \cdots, n-1$ are affine.
C_2 : The maps commute pairwise.
C_3 :For any nonempty subset A of Λ we have

$$\sigma(A) \leq \psi\{T(A)\} - \psi\left\{T\left(\overline{\text{conv}}A\right)\right\}, \tag{3.6}$$

where σ and ψ are relatively measures of non-compactness and $T = T_1T_2T_3\cdots T_{n-1}T_n$.
Then each $T_k, k = 1, 2, 3, \cdots n$ has a fixed point in Λ.

Proof.
By a similar argument as in the proof of Theorem 3.2, we get a nested sequence (Λ_n) defined as

$$\Lambda_0 = \Lambda \text{ and } \Lambda_n = \overline{\text{conv}}\{T(\Lambda_{n-1})\} \text{ for } n = 1, 2, 3, \cdots$$

Now, by inequality (3.6) we get

$$\psi\{T(\Lambda_n)\} - \psi\{T(\overline{\text{conv}})(\Lambda_{n-1})\} \geq 0, \;\; \forall\, n \in \mathbb{N}.$$

$$\implies \;\; \psi\{T(\Lambda_{n+1})\} = \psi\{T(\overline{\text{conv}})(\Lambda_n)\} \leq \psi\{T(\Lambda_n)\}, \;\; \forall\, n \in \mathbb{N}.$$

$$\implies \;\; \psi\{T(\Lambda_{n+1})\} \leq \psi\{T(\Lambda_n)\}, \;\; \forall\, n \in \mathbb{N}.$$

Thus the sequence $\psi\{T(\Lambda_n)\}$ is a nonnegative non-increasing sequence in $\mathbb{R}$ which converges to a non-negative real number say, π.

Again inequality (3.6) gives

$$\sigma(\Lambda_n) \leq \psi\{T(\Lambda_n)\} - \psi\left\{T\left(\overline{\text{conv}}(\Lambda_n)\right)\right\} = \psi\{T(\Lambda_n)\} - \psi\{T(\Lambda_{n+1})\}, \;\; \forall\, n \in \mathbb{N}.$$

$$\implies \;\; \sigma(\Lambda_n) \leq \psi\{T(\Lambda_n)\} - \psi\{T(\Lambda_{n+1})\}, \;\; \forall\, n \in \mathbb{N}.$$

Thus $\lim_{n\to\infty} \sigma(\Lambda_n) = 0.$

Finally, by virtue of the fact that Λ_n is closed for each n, $\Lambda_n = \overline{\Lambda_n}$. Hence $\lim_{n\to\infty} \sigma(\overline{\Lambda_n}) = 0.$

Now, $\Lambda_\infty = \cap_{n=1}^{\infty}\Lambda_n$, by property K_8 of Definition **2.3**. Thus $\sigma(\Lambda_\infty) \leq \sigma(\Lambda_n) \;\; \forall\, n \in \mathbb{N}$. Moreover $\sigma(\Lambda_\infty) = 0$, hence combining it with condition K_8 of Definition **2.3** results in $\overline{\Lambda_\infty} = \Lambda_\infty$ as a compact and convex set.

Since Λ_∞ is closed and compact it is invariant under $T = T_1T_2T_3\cdots T_{n-1}T_n$. In addition, $\Lambda_\infty \in \ker\sigma$, thus by Theorem **2.1** each T_k has a fixed point in Λ for $k = 1, 2, 3, \cdots, n$. □

4 CONCLUSION

Several wonderful results have emerged in the field of fixed point theory in recent times. Those that are extension of Darbos pioneering work are of special attention. Among these, none has considered a commuting family of contractions to obtain results via a measure of noncompactness. This is what has been achieved in this work. The results obtained in this article generalize, extend and improve many known results in literature specifically Theorems **2.1** and **2.4** of [5].

COMPETING INTERESTS

Author has declared that no competing interests exist.

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