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Some applications of second-order differential subordination for a class of analytic function defined by the lambda operator

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Abstract: In this paper, we introduce a new class of analytic functions by using the lambda operator and obtain some subordination results.

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1. Introduction

Let $\mathbb{C}$ be complex plane and let $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\} = \mathbb{U} \setminus \{0\}$ be an open unit disc in $\mathbb{C}$. Also let $H(\mathbb{U})$ be a class of analytic functions in $\mathbb{U}$. For $n \in \mathbb{N} = \{1, 2, 3, \cdots, \}$ and $a \in \mathbb{C}$, let $H[a, n]$ be a subclass of $H(\mathbb{U})$ formed by the functions of the form

$$f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

with $H_0 \equiv H[0, 1]$ and $H \equiv H[1, 1]$. Suppose that A_n is a class of all analytic functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_n z^n \tag{1}$$

in the open unit disk $\mathbb{U}$ with $A_1 = A$. A function $f \in H(\mathbb{U})$ is univalent if it is a one-to-one function in $\mathbb{U}$. By S, we denote a subclass of A formed by functions univalent in $\mathbb{U}$. If a function $f \in A$ maps $\mathbb{U}$ onto a convex domain and f is univalent, then f is called a convex function. By

$$K = \left\{ f \in A : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \ z \in \mathbb{U} \right\},$$

we denote a class of all convex functions defined in $\mathbb{U}$ and normalized by $f(0) = 0$ and $f'(0) = 1$.

Let f and F be elements of $H(\mathbb{U})$. A function f is said to be subordinate to F, if there exists a Schwartz function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1, \ z \in \mathbb{U}$, such that $f(z) = F(w(z))$. In this case, we write $f(z) \prec F(z)$ or $f \prec F$. Furthermore, if the function F is univalent in $\mathbb{U}$, then we get the following equivalence [1,2]:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \prec F(\mathbb{U}).$$

The method of differential subordinations (also known as the method of admissible functions) was first introduced by Miller and Mocanu in 1978 [3], and the development of the theory was originated in 1981 [4]. All details can be found in the book by Miller and Mocanu [2]. In recent years, numerous authors studied the properties of differential subordinations (see [5–8], etc.).

Let $\Psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the second-order differential subordination:

$$\Psi\left(p(z), zp'(z), zp''(z); z\right) \prec h(z), \tag{2}$$

then p is called the solution of differential subordination. The univalent function q is called a dominant of the solution of the differential subordination or, simply, a dominant if $p \prec q$ for all p satisfying (2). The dominant q_1 satisfying $q_1 \prec q$ for all dominants q of (2) is called the best dominant of (2).

Let us recall lambda function [9] defined by:

$$\lambda(z,s) = \sum_{k=2}^{\infty} \frac{z^k}{(2k+1)^k}$$

where $z \in \mathbb{U}, s \in \mathbb{C}$, when $|z| < 1, \Re(s) > 1$, when $|z| = 1$ and let $\lambda^{(-1)}(z,s)$ be defined such that

$$\lambda(z,s) * \lambda^{(-1)}(z,s) = \frac{1}{(1-z)^{\mu+1}},\ \mu > -1.$$

We now define $\left(z\lambda^{(-1)}(z,s)\right)$ as:

$$(z\lambda(z,s)) * \left(z\lambda^{(-1)}(z,s)\right) = \frac{z}{(1-z)^{\mu+1}} = z + \sum_{k=2}^{\infty} \frac{(\mu+1)_{k-1}}{(k-1)!} z^k, \mu > -1$$

and obtain the linear operator $\mathcal{I}_\mu^s f(z) = \left(z\lambda^{(-1)}(z,s)\right) * f(z)$, where $f \in A, z \in \mathbb{U}$ and $\left(z\lambda^{(-1)}(z,s)\right) = z + \sum\limits_{k=2}^{\infty} \frac{(\mu+1)_{k-1}(2k-1)^s}{(k-1)!} z^k$. A simple computation gives us

$$\mathcal{I}_\mu^s f(z) = z + \sum_{k=2}^{\infty} L(k,\mu,s) a_k z^k, \tag{3}$$

where

$$L(k,\mu,s) = \frac{(\mu+1)_{k-1}(2k-1)^s}{(k-1)!}, \tag{4}$$

where $(\mu)_k$ is the Pochhammer symbol defined in terms of the Gamma function by:

$$(\mu)_k = \frac{\Gamma(\mu+k)}{\Gamma(\mu)} = \begin{cases} 1, & \text{if } k = 0; \\ \mu(\mu+1)\cdots(\mu+k-1), & \text{if } k \in \mathbb{N}. \end{cases}.$$

Definition 1. Let $\mathfrak{L}_{\mu,s}(\varrho)$ be a class of function $f \in A$ satisfying the inequality

$$\Re\left(\mathcal{I}_\mu^s f(z)\right) \geq \varrho,$$

where $z \in \mathbb{U},\ 0 \leq \varrho < 1$ and $\mathcal{I}_\mu^s f(z)$ is the Lambda operator.

Lemma 1. *let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ be a complex number with $\Re\{\} \geq 0$. If $p \in H[a,n]$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z), \tag{5}$$

then $p(z) \prec q(z) \prec h(z)$, where $q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int\limits_0^z t^{\frac{\gamma}{n-1}} h(t)dt,\ \ z \in \mathbb{U}$. The function q is convex and is the best dominant for subordination (5).

Lemma 2. [10] *Let $\Re\{\} > 0,\ \ n \in \mathbb{N}$ and $w = \frac{n^2+|\mu|^2-|n^2-\mu^2|}{4n\Re\{\}}$. Also, let h be an analytic function in $\mathbb{U}$ with $h(0) = 1$. Suppose that $\Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w$. If $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$ and*

$$p(z) + \frac{1}{\mu} z p'(z) \prec h(z), \tag{6}$$

then $p(z) \prec q(z)$, *where* q *is a solution of the differential equation* $q(z) + \frac{n}{\mu} z q'(z) = h(z), \quad q(0) = 1$, *given by* $q(z) = \frac{\mu}{n z^{\frac{\mu}{n}}} \int\limits_0^z t^{\frac{\mu}{n-1}} h(t) dt, \; z \in \mathbb{U}$. *Moreover,* q *is the best dominant for the differential subordination* (6).

Lemma 3. [11] *Let* r *be a convex function in* $\mathbb{U}$ *and let* $h(z) = r(z) + n\varrho z r'(z), \; z \in \mathbb{U}$, *where* $\varrho > 0$ *and* $n \in \mathbb{N}$. *If* $p(z) = r(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \; z \in \mathbb{U}$, *is holomorphic in* $\mathbb{U}$ *and* $p(z) + \varrho z p'(z) \prec h(z), \; z \in \mathbb{U}$, *then* $p(z) \prec r(z)$ *and this result is sharp.*

In the present paper, we use the subordination results from [10] to prove our main results.

2. Main results

Theorem 1. *The set* $\mathfrak{L}_{\mu,s}(\varrho)$ *is convex.*

Proof. Let $f_j(z) = z + \sum\limits_{k=2}^{\infty} a_{k,j} z^k, \; z \in \mathbb{U}, \; j = 1, \cdots, m$ be in the class $\mathfrak{L}_{\mu,s}(\varrho)$. Then, by Definition 1, we get

$$\Re\left\{(\mathcal{I}_\mu^s f(z))'\right\} = \Re\left\{1 + \sum_{k=2}^{\infty} L(k,\mu,s) a_{k,j} k z^{k-1}\right\} > \varrho. \tag{7}$$

For any positive numbers $\varsigma_1, \varsigma_2, \varsigma_3, \cdots, \varsigma_m$ such that $\sum\limits_{j=1}^{m} \varsigma_j = 1$, it is necessary to show that the function $h(z) = \sum\limits_{j=1}^{m} \varsigma_j f_j(z)$ is an element of $\mathfrak{L}_{\mu,s}(\varrho)$, i.e.,

$$\Re\left\{(\mathcal{I}_\mu^s h(z))'\right\} > \varrho. \tag{8}$$

Thus, we have

$$\mathcal{I}_\mu^s h(z) = z + \sum_{k=2}^{\infty} L(k,\mu,s) \left\{\sum_{j=1}^{m} \varsigma_j a_{k,j}\right\} z^k. \tag{9}$$

If we differentiate (9) with respect to z, then we obtain

$$(\mathcal{I}_\mu^s h(z))' = 1 + \sum_{k=2}^{\infty} k L(k,\mu,s) \left\{\sum_{j=1}^{m} \varsigma_j a_{k,j}\right\} z^{k-1}.$$

Thus by using (8), we have

$$\Re\left\{(\mathcal{I}_\mu^s h(z))'\right\} = 1 + \sum_{j=1}^{m} \varsigma_j \Re\left\{\sum_{k=2}^{\infty} k L(k,\mu,s) a_{k,j} z^{k-1}\right\} > 1 + \sum_{j=1}^{m} \varsigma_j (\varrho - 1) = \varrho.$$

Hence, inequality (7) is true and we arrive at the desired result. □

Theorem 2. *Let* q *be convex function in* $\mathbb{U}$ *with* $q(0) = 1$ *and* $h(z) = q(z) + \frac{1}{\gamma+1} z q'(z), \; z \in \mathbb{U}$, *where* γ *is a complex number with* $\Re\{\gamma\} > -1$. *If* $f \in \mathfrak{L}_{\mu,s}(\varrho)$ *and* $\aleph = \Upsilon_\gamma f$, *where*

$$\aleph(z) = \Upsilon_\gamma f(z) = \frac{\gamma+1}{z^\gamma} \int\limits_0^z t^{\gamma-1} f(t) dt, \tag{10}$$

then

$$(\mathcal{I}_\mu^s f(z))' \prec h(z) \tag{11}$$

implies that $(\mathcal{I}_\mu^s \aleph(z))' \prec q(z)$ *and this result is sharp.*

Proof. In view of equality (10), we can write

$$z^{\gamma}\aleph(z) = (\gamma+1)\int_{0}^{z} t^{\gamma-1}f(t)dt. \tag{12}$$

Differentiating (12) with respect to z, we obtain $(\gamma)\aleph(z) + z\aleph'(z) = (\gamma+1)f(z)$. Further, by applying the operator $\mathcal{I}_{\mu}^{s}$ to the last equation, we get

$$(\gamma)\mathcal{I}_{\mu}^{s}\aleph(z) + z(\mathcal{I}_{\mu}^{s}\aleph(z))' = (\gamma+1)\mathcal{I}_{\mu}^{s}f(z). \tag{13}$$

If we differentiate (13) with respect to z, then we find

$$(\mathcal{I}_{\mu}^{s}\aleph(z))' + \frac{1}{\gamma+1}z(\mathcal{I}_{\mu}^{s}f(z))'' = (\mathcal{I}_{\mu}^{s}f(z))'. \tag{14}$$

By using the differential subordination given by (11) in equality (14), we obtain

$$(\mathcal{I}_{\mu}^{s}\aleph(z))' + \frac{1}{\gamma+1}z(\mathcal{I}_{\mu}^{s}f(z))'' \prec h(z). \tag{15}$$

We define

$$p(z) = (\mathcal{I}_{\mu}^{s}\aleph(z))'. \tag{16}$$

Hence, as a result of simple computations, we get

$$p(z) = \left\{z + \sum_{k=2}^{\infty} L(k,\mu,s)\frac{\gamma+1}{\gamma+k}a_k z^k\right\}' = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1,1].$$

By using (16) in subordination (15), we obtain

$$p(z) + \frac{1}{\gamma+1}zp'(z) \prec h(z) = q(z) + \frac{1}{\gamma+1}zq'(z), \quad z \in \mathbb{U}.$$

If we use Lemma 2, then we write $p(z) \prec q(z)$. Thus, we obtained the desired result and q is the best dominant. □

Example 1. If we choose $\gamma = i+1$ and $q(z) = \frac{1+z}{1-z}$, in Theorem 2, then we get $h(z) = \frac{(i+2)-((i+2)z+2)z}{(i+2)(1-z)^2}$. If $f \in \mathfrak{L}_{\mu,s}(\varrho)$ and $\aleph$ is given as $\aleph(z) = \Upsilon_i f(z) = \frac{i+2}{z^{i+1}}\int_{0}^{z} t^i f(t)dt$, then, by virtue of Theorem 2, we find $(\mathcal{I}_{\mu}^{s}f(z))' \prec h(z) = \frac{(i+2)-((i+2)z+2)z}{(i+2)(1-z)^2}$, implies $(\mathcal{I}_{\mu}^{s}f(z))' \prec \frac{1+z}{1-z}$.

Theorem 3. *Let $\Re\{\gamma\} > -1$ and $w = \frac{1+|\gamma+1|^2-|\gamma^2+2\gamma|}{4\Re\{\gamma+1\}}$. Suppose that h is an analytic function in $\mathbb{U}$ with $h(0) = 1$ and that $\Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w$. If $f \in \mathfrak{L}_{\mu,s}(\varrho)$ and $\aleph = \Upsilon_{\mu}^{s}f$, where $\aleph$ is defined by (10), then*

$$(\mathcal{I}_{\mu}^{s}f(z))' \prec h(z) \tag{17}$$

implies that $(\mathcal{I}_{\mu}^{s}\aleph(z))' \prec q(z)$, where q is the solution of the differential equation $h(z) = q(z) + \frac{1}{\gamma+1}zq'(z),\ q(0) = 1$, given by $q(z) = \frac{\gamma+1}{z^{\gamma+1}}\int_{0}^{z} t^{\gamma}f(t)dt$. Moreover, q is the best dominant for subordination (17).

Proof. If we choose $n = 1$ and $\mu = \gamma+1$ in Lemma 1, then the proof is obtained by means of the proof of Theorem 3. □

Theorem 4. *Let*

$$h(z) = \frac{1+(2\varrho-1)z}{1+z}, \quad 0 \le \varrho < 1 \tag{18}$$

be convex in $\mathbb{U}$ *with* $h(0) = 1$. *If* $f \in A$ *and verifies the differential subordination* $(\mathcal{I}_\mu^s f(z))' \prec h(z)$, *then* $(\mathcal{I}_\mu^s \aleph(z))' \prec q(z) = (2\varrho - 1) + \frac{2(1-\varrho)(\gamma+1)\tau(\gamma)}{z^{\gamma+1}}$, *where* τ *is given by the formula*

$$\tau(\gamma) = \int_0^z \frac{t^\gamma}{t+1} dt \tag{19}$$

and $\aleph$ *is given by equation (10). The function* q *is convex and is the best dominant.*

Proof. If $h(z) = \frac{1+(2\varrho-1)z}{1+z}$, $0 \le \varrho < 1$, then h is convex and, in view of Theorem 3, we can write $(\mathcal{I}_\mu^s \aleph(z))' \prec q(z)$. Now, by using Lemma1, we get

$$q(z) = \frac{\gamma+1}{z^{\gamma+1}} \int_0^z t^\gamma h(t) dt = \frac{\gamma+1}{z^{\gamma+1}} \int_0^z t^\gamma \left\{ \frac{1+(2\varrho-1)t}{1+t} \right\} dt = (2\varrho - 1) + \frac{2(1-\varrho)(\gamma+1)}{z^{\gamma+1}} \tau(\gamma),$$

where τ is given by (19). Hence, we obtain

$$(\mathcal{I}_\mu^s \aleph(z))' \prec q(z) = (2\varrho - 1) + \frac{2(1-\varrho)(\gamma+1)\tau(\gamma)}{z^{\gamma+1}}.$$

The function q is convex. Moreover, it is the best dominant. Hence the theorem is proved. □

Theorem 5. *If* $0 \le \varrho < 1, 0 \le \mu < 1, \delta \ge 0, \Re\{\gamma\} > -1$, *and* $\aleph = \Upsilon_\gamma f$ *is defined by (10), then* $\Upsilon_\gamma(\mathfrak{L}_{\mu,s}(\varrho)) \subset \mathfrak{L}_{\mu,s}(\rho)$, *where*

$$\rho = \min_{|z|=1} \Re\{q(z)\} = \rho(\gamma, \varrho) = (2\varrho - 1) + 2(1-\varrho)(\gamma+1)\tau(\gamma) \tag{20}$$

and τ *is given by (19).*

Proof. Assume that h is given by equation (18), $f \in \mathfrak{L}_{\mu,s}(\varrho)$, and $\aleph = \Upsilon_\gamma f$ is defined by (10). Then h is convex and, by Theorem 3, we deduce

$$(\mathcal{I}_\mu^s \aleph(z))' \prec q(z) = (2\varrho - 1) + \frac{2(1-\varrho)(\gamma+1)\tau(\gamma)}{z^{\gamma+1}}, \tag{21}$$

where τ is given by (19). Since q is convex, $q(\mathbb{U})$ is symmetric about the real axis, and $\Re\{\gamma\} > -1$, we find

$$\Re\left\{(\mathcal{I}_\mu^s \aleph(z))'\right\} \ge \min_{|z|=1} \Re\{q(z)\} = \Re\{q(1)\} = \rho(\gamma, \varrho) = (2\varrho - 1) + 2(1-\varrho)(\gamma+1)(1-\varrho)\tau(\gamma).$$

It follows from inequality (21) that $\Upsilon_\gamma(\mathfrak{L}_{\mu,s}(\varrho)) \subset \mathfrak{L}_{\mu,s}(\rho)$, where ρ is given by (20). Hence the theorem is proved. □

Theorem 6. *Let* q *be a convex function with* $q(0) = 1$ *and* h *be a function such that* $h(z) = q(z) + zq'(z)$, $z \in \mathbb{U}$. *If* $f \in A$, *then the subordination*

$$(\mathcal{I}_\mu^s f(z))' \prec h(z) \tag{22}$$

implies that $\frac{\mathcal{I}_\mu^s f(z)}{z} \prec q(z)$, *and the result is sharp.*

Proof. Let

$$p(z) = \frac{\mathcal{I}_\mu^s f(z)}{z}. \tag{23}$$

Differentiating (23), we find $(\mathcal{I}_\mu^s f(z))' = p(z) + zp'(z)$. We now compute $p(z)$. This gives

$$p(z) = \frac{\mathcal{I}_\mu^s f(z)}{z} = \frac{z + \sum_{k=2}^{\infty} L(k,\mu,s) a_k z^k}{z} = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1,1]. \tag{24}$$

By using (24) in subordination (22), we find $p(z) + zp'(z) \prec h(z) = q(z) + zq'(z)$. Hence, by applying Lemma 3, we conclude that $p(z) \prec q(z)$ i.e., $\frac{\mathcal{I}_{\mu}^{s} f(z)}{z} \prec q(z)$. This result is sharp and q is the best dominant. Hence the theorem is proved. □

Example 2. If we take $\mu = 0$ and $s = 1$ in equality (4) and $q(z) = \frac{1}{1-z}$ in Theorem 5, then $h(z) = \frac{1}{(1-z)^2}$ and

$$I_0^1 f(z) = z + \sum_{k=2}^{\infty} \frac{(2k-1)}{(k-1)!} a_k z^k. \tag{25}$$

Differentiating (25) with respect to z, we get

$$(I_0^1 f(z))' = 1 + \sum_{k=2}^{\infty} \frac{(2k-1)}{(k-1)!} a_k z^{k-1} = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1,1].$$

By using Theorem 5, we find $(I_0^1 f(z))' \prec h(z) = \frac{1}{(1-z)^2}$. This yields $\frac{I_0^1 f(z)}{z} \prec q(z) = \frac{1}{1-z}$.

Theorem 7. *Let $h(z) = \frac{1+(2\varrho-1)z}{1+z}$, $z \in \mathbb{U}$ be convex in $\mathbb{U}$ with $h(0) = 1$ and $0 \leq \varrho < 1$. If $f \in A$ satisfies the differential subordination*

$$(\mathcal{I}_{\mu}^{s} f(z))' \prec h(z), \tag{26}$$

then $\frac{\mathcal{I}_{\mu}^{s} f(z)}{z} \prec q(z) = (2\varrho - 1) + \frac{2(1-\varrho)ln(1+z)}{z}$. The function q is convex and, in addition, it is the best dominant.

Proof. Let

$$p(z) = \frac{\mathcal{I}_{\mu}^{s} f(z)}{z} = 1 + p_1 z + p_2 z^2 + \cdots, \quad p \in H[1,1]. \tag{27}$$

Differentiating (27), we find

$$(\mathcal{I}_{\mu}^{s} f(z))' = p(z) + zp'(z). \tag{28}$$

In view of (28), the differential subordination (26) becomes $(\mathcal{I}_{\mu}^{s} f(z))' \prec h(z) = \frac{1+(2\varrho-1)z}{1+z}$, and by using Lemma 1, we deduce $p(z) \prec q(z) = \frac{1}{z} \int h(t)dt = (2\varrho - 1) + \frac{2(1-\varrho)ln(1+z)}{z}$. Now, by virtue of relation (27) we obtained the desired result. □

Corollary 1. *If $f \in \mathfrak{L}_{\mu,s}(\varrho)$, then $\Re\left(\frac{\mathcal{I}_{\mu}^{s} f(z)}{z}\right) > (2\varrho - 1) + 2(1-\varrho)ln(2)$.*

Proof. If $f \in \mathfrak{L}_{\mu,s}(\varrho)$, then it follows from Definition 1 that $\Re\left\{(\mathcal{I}_{\mu}^{s} f(z))'\right\} > \varrho, \; z \in \mathbb{U}$, which is equivalent to $(\mathcal{I}_{\mu}^{s} f(z))' \prec h(z) = \frac{1+(2\varrho-1)z}{1+z}$. Now, by using Theorem 7, we obtain

$$\frac{\mathcal{I}_{\mu}^{s} f(z)}{z} \prec q(z) = (2\varrho - 1) + \frac{2(1-\varrho)ln(1+z)}{z}.$$

Since q is convex and $q(\mathbb{U})$ is symmetric about the real axis, we conclude that

$$\Re\left(\frac{\mathcal{I}_{\mu}^{s} f(z)}{z}\right) > \Re(q(1)) = (2\varrho - 1) + 2(1-\varrho)ln(2).$$

□

Theorem 8. *Let q be a convex function such that $q(0) = 1$ and h be the function given by the formula $h(z) = q(z) + zq'(z)$, $z \in \mathbb{U}$. If $f \in A$ and verifies the differential subordination*

$$\left\{\frac{z\mathcal{I}_{\mu}^{s} f(z)}{\mathcal{I}_{\mu}^{s} \aleph(z)}\right\}' \prec h(z), \quad z \in \mathbb{U}, \tag{29}$$

then $\frac{\mathcal{I}_\mu^s f(z)}{\mathcal{I}_\mu^s \aleph(z)} \prec q(z)$, $z \in \mathbb{U}$, *and this result is sharp.*

Proof. For function $f \in A$, given by Equation (1), we get

$$\mathcal{I}_\mu^s \aleph(z) = z + \sum_{k=2}^{\infty} L(k,\mu,s)\frac{\gamma+1}{k+\gamma} a_k b_k z^k, \ \ z \in \mathbb{U}.$$

We now consider the function

$$p(z) = \frac{\mathcal{I}_\mu^s f(z)}{\mathcal{I}_\mu^s \aleph(z)} = \frac{z + \sum\limits_{k=2}^{\infty} L(k,\mu,s) a_k b_k z^k}{z + \sum\limits_{k=2}^{\infty} L(k,\mu,s)\frac{\gamma+1}{k+\gamma} a_k b_k z^k} = \frac{1 + \sum\limits_{k=2}^{\infty} L(k,\mu,s) a_k b_k z^{k-1}}{1 + \sum\limits_{k=2}^{\infty} L(k,\mu,s)\frac{\gamma+1}{k+\gamma} a_k b_k z^{k-1}}.$$

In this case, we get

$$(p(z))' = \frac{(\mathcal{I}_\mu^s f(z))'}{\mathcal{I}_\mu^s \aleph(z)} - p(z)\frac{(\mathcal{I}_\mu^s \aleph(z))'}{\mathcal{I}_\mu^s \aleph(z)}.$$

Then

$$p(z) + zp'(z) = \left\{ \frac{z\mathcal{I}_\mu^s f(z)}{\mathcal{I}_\mu^s \aleph(z)} \right\}', \ \ z \in \mathbb{U}. \tag{30}$$

By using relation (30) in inequality (29), we obtain $p(z) + zp'(z) \prec h(z) = q(z) + zq'(z)$ and, by virtue of Lemma 3, $p(z) \prec q(z)$, i.e., $\frac{\mathcal{I}_\mu^s f(z)}{\mathcal{I}_\mu^s \aleph(z)} \prec q(z)$. Hence the theorem is proved. □

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