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On the Solvability of an Evolution Problem with Weighted Integral Boundary Conditions in Sobolev Function Spaces with a Priori Estimate and Fourier's Method

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Abstract

Aims/ The aims of this paper are to prove existence and uniqueness of following integral boundary conditions mixed problem for parabolic equation :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{a(t)}{x^{n+1}}\dfrac{\partial}{\partial x}\left(x^{n+3}\dfrac{\partial u}{\partial x}\right) + bu = \vartheta(x,t), \\ u(x,0) = \lambda(x),\ 0 \le x \le \ell, \\ \displaystyle\int_0^\ell x^{n-1}u(x,t)\,dx = E_n(t),\ 0 \le t \le T, \\ \displaystyle\int_0^\ell x^{n}u(x,t)\,dx = G_n(t),\ 0 \le t \le T. \end{cases}$$

The proofs are based on a priori estimates established in Sobolev function spaces and Fourier's method.

Keywords: Fourier's method; A priori Estimate; Nonlocal conditions; Mixed Problem; Parabolic Equation; Sobolev Espace.

2000 Mathematics Subject Classification: 35K20; 35B30; 35D05; 46E40; 46E99.

1 Introduction

This paper deals with existence and uniqueness of a following class of parabolic equation with time and space-variable characteristics :

$$\frac{\partial u}{\partial t} - \frac{a(t)}{x^{n+1}}\frac{\partial}{\partial x}\left(x^{n+3}\frac{\partial u}{\partial x}\right) + bu = \vartheta(x,t), \tag{1.1}$$

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satisfying the initial condition

$$u(x,0) = \lambda(x),\ 0 \le x \le \ell, \tag{1.2}$$

and the integral conditions

$$\int_0^\ell x^{n-1} u(x,t)\,dx = E_n(t),\ 0 \le t \le T, \tag{1.3}$$

$$\int_0^\ell x^{n} u(x,t)\,dx = G_n(t),\ 0 \le t \le T, \tag{1.4}$$

where $\lambda,\ E_n,\ G_n,\ a$ and ϑ are known functions, and b, ℓ, T are positive constants.

Condition 1.1. *For all* $(x,t) \in \overline{\Omega}$, *we assume that*

$$a_0 \le a(t) \le a_1,$$

$$a_2 \le \frac{da(t)}{dt} \le a_3,$$

$$a_4 \le a(t) - x\frac{da(t)}{dt} \le a_5,$$

where $a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5$ *are positive constants.*

The data satisfies the following compatibility conditions : for consistency, we have

$$\int_0^\ell x^n \lambda(x)\,dx = E_n(0), \quad \text{and} \quad \int_0^\ell x^{n+1}\lambda(x)\,dx = G_n(0).$$

The importance of problems with integral conditions has been pointed out by Samarskii[1]. Mathematical modelling by evolution problems with a nonlocal constraint of the form $\dfrac{1}{1-\ell}\displaystyle\int_\ell^1 u(x,t)\,dx = \zeta(t)$ is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physic.

Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. [2] used the potentiel method, combining a Dirichlet and an intégral condition for a parabolic equation. [3] used the maximum principle, combining a Neumann and an integral condition for heat equation. [4] and [5] used the Fourier method for same purpose.
Recently, mixed problems with integral conditions for generalization of equation (1.1) have been treated using the energy-integral method. See and [6], [7],[8], [9], [10], [11],[12], [13],[14],[15]. Differently to these works, in the present paper we combine a priori estimate and Fourier's method to prove existence and uniqueness solution of the problem (1.1)- (1.4).
The results obtained in this paper generalize the results of [5], and constitute a new contribution to this emerging field of research . It is interesting to note that the application of Fourier method to this nonlocal problem is made possible thanks, essentially, to the use of a Sobolev function space.

To this, we reduce the inhomogeneous boundary conditions (1.3) and (1.4) to homogeneous conditions, by introducing a new unknown function v by $v(x,t) = u(x,t) - w(x,t)$, where

$$w(x,t) = \frac{-(n+3)x + (n+1)\ell}{\ell^{n+3}} E_n(t) + \frac{(n+3)x - (n+1)\ell}{\ell^{n+3}} G_n(t). \tag{1.5}$$

Then, problem (1.1), (1.2), (1.3) and (1.4) is transformed into the following homogeneous boundary value problem

$$\frac{\partial v}{\partial t}-\frac{a(t)}{x^{n+1}}\frac{\partial}{\partial x}\left(x^{n+3}\frac{\partial v}{\partial x}\right)+bv=\xi(x,t),\quad (x,t)\in\Omega, \tag{1.6}$$

$$v(x,0)=\Lambda(x),\ 0\le x\le \ell, \tag{1.7}$$

$$\int_0^\ell x^n v(x,t)\,dx=0,\ 0\le t\le T, \tag{1.8}$$

$$\int_0^\ell x^{n+1} v(x,t)\,dx=0,\ 0\le t\le T, \tag{1.9}$$

where

$$\begin{aligned}\xi(x,t)=&\vartheta(x,t)-\frac{(n+3)^2xa(t)}{\ell^{n+3}}E_n(t)+\frac{(n+3)x-(n+1)\ell}{\ell^{n+3}}(bE_n(t)+E_n'(t))\\&+\frac{(n+3)^2xa(t)}{\ell^{n+3}}G_n(t)-\frac{(n+3)x-(n+1)\ell}{\ell^{n+3}}(bG_n(t)+G_n'(t)),\end{aligned}$$

$$\Lambda(x)=\lambda(x)+\frac{(n+3)x-(n+3)\ell}{\ell^{n+3}}E_n(0)+\frac{(n+3)x-(n+1)\ell}{\ell^{n+3}}G_n(0).$$

Here, we assume that the function Λ satisfy conditions of (1.8) and (1.9), that is

$$\int_0^\ell x^n\Lambda(x)\,dx=\int_0^\ell x^{n+1}\Lambda(x)\,dx=0. \tag{1.10}$$

Instead of searching for the function u, we search for the function v. So the solution of problem (1.1), (1.2), (1.3) and (1.4) will be given by $u(x,t)=v(x,t)+w(x,t)$.

The general difficult which arises to us is the presence of integral conditions which complicates the application of standard methods. It may, however, be worth while if this type of problem can be transformed into another equivalent problem which involves no integral conditions. For this, we convert problem (1.6), (1.7), (1.8) and (1.9) to the following classical problem.

Theorem 1.2. *The problem* (1.6)*,* (1.7)*,* (1.8) *and* (1.9) *is equivalent to the following classical problem :*

$$\frac{\partial v}{\partial t}-\frac{a(t)}{x^{n+1}}\frac{\partial}{\partial x}\left(x^{n+3}\frac{\partial v}{\partial x}\right)+bv=\xi(x,t),\quad (x,t)\in\Omega, \tag{1.11}$$

$$v(x,0)=\Lambda(x),\ 0\le x\le \ell, \tag{1.12}$$

$$v(\ell,t)=\frac{1}{\ell a(t)}\int_0^\ell (x-\ell)x^n\xi(x,t)\,dx,\ 0\le t\le T, \tag{1.13}$$

$$\frac{\partial v}{\partial x}(\ell,t)=-\frac{1}{\ell^{n+3}a(t)}\int_0^\ell x^{n+1}\xi(x,t)\,dx,\ 0\le t\le T. \tag{1.14}$$

Proof
Multiplying (1.6) with x^n, and integrating the obtained result with respect x over $(0,\ell)$, we obtain

$$\frac{\partial}{\partial t}\int_0^\ell x^n v\,dx - a(t)\int_0^\ell \frac{1}{x}\frac{\partial}{\partial x}\left(x^{n+3}\frac{\partial v}{\partial x}\right)\,dx + b\int_0^\ell x^n v\,dx = \int_0^\ell x^n \xi(x,t)\,dx. \tag{1.15}$$

Integrating by parts the integrals on the left-hand side of (1.15), and taking into account condition (1.8), we get

$$\ell^{n+2}\frac{\partial v}{\partial x}(\ell,t) + \ell^{n+1}v(\ell,t) = -\frac{1}{a(t)}\int_0^\ell x^n\xi(x,t)\,dx. \tag{1.16}$$

Multiplying (1.6) with x^{n+1} and integrating the result obtained over $(0,\ell)$, he have

$$-a(t)\int_0^\ell \frac{\partial}{\partial x}\left(x^{n+3}\frac{\partial v}{\partial x}\right)\,dx = \int_0^\ell x^{n+1}\xi(x,t)\,dx. \tag{1.17}$$

Integrating by parts the integrals on the left-hand side of (1.17).

$$\frac{\partial v}{\partial x}(\ell,t) = -\frac{1}{\ell^{n+3}a(t)}\int_0^\ell x^{n+1}\xi(x,t)\,dx \tag{1.18}$$

Combining the equalities (1.17) and (1.18), we have

$$v(\ell,t) = \frac{1}{\ell a(t)}\int_0^\ell (x-\ell)x^n\xi(x,t)\,dx. \tag{1.19}$$

It remains to prove that $\int_0^\ell x^n v(x,t)\,dx = 0$ and $\int_0^\ell x^{n+1}v(x,t)\,dx = 0.$
By using (1.6) and taking into account (1.16) and (1.18) we get

$$\frac{d}{dt}\int_0^\ell x^n v(x,t)\,dx + b\int_0^\ell x^n v(x,t)\,dx = 0,\ 0\le t\le T$$

$$\frac{d}{dt}\int_0^\ell x^{n+1} v(x,t)\,dx + b\int_0^\ell x^{n+1} v(x,t)\,dx = 0,\ 0\le t\le T$$

By virtue of the compatibility of the conditions, it follows that

$$\int_0^\ell x^n v(x,t)\,dx = \int_0^\ell x^{n+1}v(x,t)\,dx = 0.$$

This complete the proof of Theorem (1.2). ⊡.

By introducing the new unknown function

$$z(x,t) = v(x,t) - \theta_n(x,t)\int_0^\ell x^{n+1}\xi(x,t)\,dx - \eta_n(x,t)\int_0^\ell x^n\xi(x,t)\,dx,$$

where

$$\theta_n(x,t) = \frac{\ell^{n+4}a(t)x^2 + (1-2\ell^{n+5})a(t)x + \ell(\ell^{n+5}a(t)-1)}{\ell^{n+3}a(t)}$$

and

$$\eta_n(x,t) = \frac{\ell^2 a(t)x^2 - 2\ell^3 a(t)x + \ell^4 a(t) + 1}{\ell a(t)},$$

the problem (1.11), (1.12), (1.13) and (1.14) is transformed into the following homogeneous local boundary conditions problem,

$$\frac{\partial z}{\partial t} - \frac{a(t)}{x^{n+1}}\frac{\partial}{\partial x}\left(x^{n+3}\frac{\partial z}{\partial x}\right) + bz = f(x,t), \quad (x,t) \in \Omega, \tag{1.20}$$

$$z(x,0) = \varphi(x),\ 0 \le x \le \ell, \tag{1.21}$$

$$z(\ell,t) = 0,\ 0 \le t \le T, \tag{1.22}$$

$$\frac{\partial z}{\partial x}(\ell,t) = 0,\ 0 \le t \le T, \tag{1.23}$$

where

$$\begin{aligned} f(x,t) =& \xi(x,t) - \frac{\ell^{n+4}a'(t)x^2 + (1-2\ell^{n+5}a'(t) + \ell(\ell^{n+5}a'(t)-1)}{\ell^{n+3}a'(t)}\int_0^\ell x^{n+1}\frac{\partial \xi}{\partial t}(x,t)\,dx \\ &- \frac{\ell^2 a'(t)x^2 - 2\ell^3 a'(t)x + \ell^4 a'(t) + 1}{\ell a'(t)}\int_0^\ell x^n \frac{\partial \xi}{\partial t}(x,t)\,dx \\ &- \frac{\ell^{n+4}a(t)x^2 + (1-2\ell^{n+5})a(t)x + \ell(\ell^{n+5}a(t)-1)}{\ell^{n+3}a(t)} b(t)\int_0^\ell x^{n+1}\xi(x,t)\,dx \\ &- \frac{\ell^2 a(t)x^2 - 2\ell^3 a(t)x + \ell^4 a(t) + 1}{\ell a(t)} b(t)\int_0^\ell x^n \xi(x,t)\,dx \\ &+ 2\ell x a(t)[(n+4)x - n - 3]\int_0^\ell x^n \xi(x,t)\,dx \\ &+ \frac{2(n+4)\ell^{n+4}a(t)x^2 + (3+n)(1-2\ell^{n+5})a(t)x}{\ell^{n+3}}\int_0^\ell x^{n+1}\xi(x,t)\,dx, \end{aligned}$$

$$\begin{aligned} \varphi(x) =& \Lambda(x) - \frac{\ell^{n+4}a(0)x^2 + (1-2\ell^{n+5}a(0) + \ell(\ell^{n+5}a(0)-1)}{\ell^{n+3}a(0)}\int_0^\ell x^{n+1}\xi(x,0)\,dx \\ &- \frac{\ell^2 a(0)x^2 - 2\ell^3 a(0)x + \ell^4 a(0) + 1}{\ell a(0)}\int_0^\ell x^n \xi(x,0)\,dx. \end{aligned}$$

The rest of this paper is organized as follows : in section 2, we establish a priori estimate. Finally, in section 3, we prove existence of generalized solution.

2 An priori estimate

The problem (1.20), (1.21), (1.22) and (1.23) can be considered as solving the following operator equation :

$$Az = (\varphi, f) = \mathcal{F},$$

where A is an operator defined on $\mathbb{E}$ into $\mathbb{F}$. $\mathbb{F}$ is the banach space of functions $z \in L^2(\Omega)$, satisfying conditions (1.22) and (1.23) with the norm

$$\|z\|_{\mathbb{E}}^2 = \int_{\Omega_\tau} \left(x^n z^2 + x^{n+1} \left| \frac{\partial z}{\partial x} \right|^2 + x^{n+1} \left| \frac{\partial z}{\partial t} \right|^2 \right) dxdt + \sup_{0 \le t \le T} \int_0^\ell \left(x^{n+1} z^2 + x^{n+3} \left| \frac{\partial z}{\partial x} \right|^2 \right) dx$$

and $\mathbb{F}$ is the Hilbert space $L^2(\Omega) \times L^2(0, \ell)$ which consists of elements $\mathcal{F} = (f,\ \varphi)$ with the norm

$$\|\mathcal{F}\|_{\mathbb{F}}^2 = \int_0^\ell \varphi^2(x)\, dx + \int_0^\ell \left(\frac{d\varphi}{dx} \right)^2 dx + \int_{\Omega_\tau} f^2(x,t)\, dxdt.$$

Let $D(A)$ be the set of all function z, for which $z,\ x^{n+1}z,\ x^{n+3}\frac{\partial z}{\partial x} \in L^2(0,\ell)$ and $z,\ x^{n+1}\frac{\partial z}{\partial x}$, $x^{n+1}\frac{\partial z}{\partial t},\ x^{n+3}\frac{\partial z}{\partial x},\ \frac{\partial}{\partial x}\left(x^{n+3}\frac{\partial z}{\partial x} \right) \in L^2(\Omega)$.

Theorem 2.1. *For any function $z \in D(A)$, we have*

$$\|z\|_{\mathbb{E}} \le C \|Az\|, \tag{2.1}$$

where $C = \sqrt{\dfrac{\max((1+b)\ell^{n+1},\ a_1\ell^{n+3})}{\min(2b + (n+1)a_1 - 1, a_4\ell, 1+b, a_0\ell^2)}}$.

Proof

Multiplying the equation (1.20) with $x^n z(x,t) + x^{n+1}\frac{\partial z}{\partial t}(x,t)$ and integrating the results obtained over $\Omega_\tau = (0,\ell) \times (0,T)$. Observe that

$$\begin{aligned} &\int_{\Omega_\tau} x^n z \frac{\partial z}{\partial t}\, dxdt + \int_{\Omega} x^{n+1} \left(\frac{\partial z}{\partial t} \right)^2 dxdt - \int_{\Omega_\tau} \frac{a(t)z}{x} \frac{\partial}{\partial x} \left(x^{n+3} \frac{\partial z}{\partial x} \right) dxdt \\ &- \int_{\Omega_\tau} a(t) \frac{\partial}{\partial x} \left(x^{n+3} \frac{\partial z}{\partial x} \right) \frac{\partial z}{\partial t}\, dxdt + \int_{\Omega_\tau} b x^n z^2\, dxdt + \int_{\Omega_\tau} b x^{n+1} z \frac{\partial z}{\partial t}\, dxdt \\ &= \int_{\Omega_\tau} x^n z f(x,t)\, dxdt + \int_{\Omega_\tau} x^{n+1} f(x,t) \frac{\partial z}{\partial t}\, dxdt. \end{aligned} \tag{2.2}$$

Integrating by parts the terms of left-hand side of (2.2), we get

$$\int_{\Omega_\tau} x^n z \frac{\partial z}{\partial t}\, dxdt = \frac{1}{2} \int_0^\ell x^{n+1} z^2(x,\tau)\, dx - \frac{1}{2} \int_0^\ell x^{n+1} \varphi^2(x)\, dx, \tag{2.3}$$

$$\begin{aligned} - \int_{\Omega_\tau} \frac{a(t)z}{x} \frac{\partial}{\partial x} \left(x^{n+3} \frac{\partial z}{\partial x} \right) dxdt =& \frac{n+1}{2} \int_{\Omega_\tau} a(t) x^n z^2\, dxdt \\ &+ \frac{1}{2} \int_{\Omega_\tau} a(t) x^{n+2} \left(\frac{\partial z}{\partial x} \right)^2 dxdt, \end{aligned} \tag{2.4}$$

$$\int_{\Omega_\tau} b x^{n+1} z \frac{\partial z}{\partial t}\, dxdt = \frac{1}{2} \int_0^\ell b x^{n+1} z^2(x,\tau)\, dx - \frac{1}{2} \int_0^\ell b x^{n+1} \varphi^2(x)\, dx, \tag{2.5}$$

$$\begin{aligned} - \int_{\Omega_\tau} a(t) \frac{\partial}{\partial x} \left(x^{n+3} \frac{\partial z}{\partial x} \right) \frac{\partial z}{\partial t}\, dxdt = \frac{1}{2} \int_0^\ell a(t) x^{n+3} \left(\frac{\partial z}{\partial x} \right)^2 dx \\ - \frac{1}{2} \int_0^\ell a(t) x^{n+3} \left(\frac{d\varphi}{dx} \right)^2 dx - \frac{1}{2} \int_{\Omega_\tau} a'(t) x^{n+3} \left(\frac{\partial z}{\partial x} \right)^2 dxdt. \end{aligned} \tag{2.6}$$

By substituting(2.3), (2.4), (2.5) and (2.6) into (2.2), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_\tau}(2b+(n+1)a(t))x^n z^2\,dxdt+\frac{1}{2}\int_{\Omega_\tau}\left(a(t)-x\frac{da(t)}{dt}\right)x^{n+2}\left(\frac{\partial z}{\partial x}\right)^2 dxdt\\
&+\int_{\Omega_\tau}x^{n+1}\left(\frac{\partial z}{\partial t}\right)^2 dxdt+\frac{1+b}{2}\int_0^\ell x^{n+1}z^2(x,\tau)\,dxdt+\frac{1}{2}\int_0^\ell a(t)x^{n+3}\left(\frac{\partial z}{\partial x}\right)^2 dx\\
&=\frac{1+b}{2}\int_0^\ell x^{n+1}\varphi^2(x)\,dx+\frac{1}{2}\int_0^\ell a(t)x^{n+3}\left(\frac{d\varphi}{dx}\right)^2 dx+\int_{\Omega_\tau}x^n z f(x,t)\,dxdt\\
&+\int_{\Omega_\tau}x^{n+1}f(x,t)\frac{\partial z}{\partial t}\,dxdt. \qquad (2.7)
\end{aligned}$$

Estimating the two last integrals of the right-hand side of (2.7), by applying elementary inequalities, we get

$$\int_{\Omega_\tau}x^n z f(x,t)\,dxdt\le\frac{1}{2}\int_{\Omega_\tau}x^n z^2\,dxdt+\frac{1}{2}\int_{\Omega_\tau}x^n f^2(x,t)\,dxdt, \qquad (2.8)$$

$$\int_{\Omega_\tau}x^{n+1}\frac{\partial z}{\partial t}f(x,t)\,dxdt\le\frac{1}{2}\int_{\Omega_\tau}x^{n+1}\left(\frac{\partial z}{\partial t}\right)^2 dxdt+\frac{1}{2}\int_{\Omega_\tau}x^{n+1}f^2(x,t)\,dxdt. \qquad (2.9)$$

Therefore, by formulas (2.7), (2.8) and (2.9), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_\tau}(2b+(n+1)a(t)-1)x^n z^2\,dxdt+\frac{1}{2}\int_{\Omega_\tau}\left(a(t)-x\frac{da(t)}{dt}\right)x^{n+2}\left(\frac{\partial z}{\partial x}\right)^2 dxdt\\
&+\frac{1}{2}\int_{\Omega_\tau}x^{n+1}\left(\frac{\partial z}{\partial t}\right)^2 dxdt+\frac{1+b}{2}\int_0^\ell x^{n+1}z^2(x,\tau)\,dxdt+\frac{1}{2}\int_0^\ell a(t)x^{n+3}\left(\frac{\partial z}{\partial x}\right)^2 dx\\
&\le\frac{1+b}{2}\int_0^\ell x^{n+1}\varphi^2(x)\,dx+\frac{1}{2}\int_0^\ell a(t)x^{n+3}\left(\frac{d\varphi}{dx}\right)^2 dx+\frac{1}{2}\int_{\Omega_\tau}(1+x)x^n f^2(x,t)\,dxdt. \qquad (2.10)
\end{aligned}$$

Taking account the assumptions 1.1, from (2.10), it follows that

$$\begin{aligned}
&\int_{\Omega_\tau}x^n z^2\,dxdt+\int_{\Omega_\tau}x^{n+2}\left(\frac{\partial z}{\partial x}\right)^2 dxdt+\int_{\Omega_\tau}x^{n+1}\left(\frac{\partial z}{\partial t}\right)^2 dxdt+\int_0^\ell x^{n+1}z^2(x,\tau)\,dxdt\\
&+\int_0^\ell x^{n+3}\left(\frac{\partial z}{\partial x}\right)^2 dx\le M\left\{\int_0^\ell\varphi^2(x)\,dx+\int_0^\ell\left(\frac{d\varphi}{dx}\right)^2 dx+\int_{\Omega_\tau}f^2(x,t)\,dxdt\right\}, \qquad (2.11)
\end{aligned}$$

where $M=\dfrac{\max((1+b)\ell^{n+1},\ a_1\ell^{n+3})}{\min(2b+(n+1)a_1-1,a_4\ell,1+b,a_0\ell^2)}$.

The right-hand side of (2.11) is independent of τ, replacing the left-hand side by the upper with respect to τ. Thus inequality (2.1) holds, where

$$C=\sqrt{\frac{(\max((1+b)\ell^{n+1},\ a_1\ell^{n+3})}{\min(2b+(n+1)a_1-1,\ a_4\ell,1+b,a_0\ell^2)}}.$$

This completes the proof of Theorem (1.2). ⊡

3 Solvability of the problem

Now we shall start to prove the existence of the boundary value problem (1.20), (1.21), (1.22) and (1.23). We use the Fourier's method.

Consider the function $z_m(x,t) = y_m(x)w_m(t)$, where $y_m(t)$ is a eigenfunction of the following boundary value problem

$$\begin{cases} -\dfrac{a(t)}{x^{n+1}}\dfrac{d}{dx}\left(x^{n+3}\dfrac{dy_m(x)}{dx}\right) + by_m(x) = \beta_m y_m(x), \\ y_m(\ell) = 0, \\ \dfrac{dy_m(\ell)}{dx} = 0, \end{cases}$$

where β_m is the eigenvalue corresponding to the eigenfunction $y_m(x)$, and $w_n(t)$ satisfying the initial problem

$$\begin{cases} \dfrac{dw_m(t)}{dt} - \alpha_m w_m(t) = f_m(t), \\ w_m(0) = \varphi_m. \end{cases}$$

Here

$$\varphi(x) = \sum_{m=1}^{+\infty} \varphi_m y_m(x),$$
$$\varphi'(x) = \sum_{m=0}^{+\infty} \rho_m y_m(x),$$
$$f(x,t) = \sum_{m=1}^{+\infty} f_m(t) y_m(x).$$

Using the Parseval-Steklov equality, we have

$$\|(f,\varphi)\|_{\mathbb{F}} = \sum_{m=1}^{+\infty}\left(\int_0^T f_m^2(t)\,dt + \varphi_m^2 + \rho_m^2\right).$$

The direct computation, the solution of the initial problem is giving by

$$w_m(t) = \varphi_m e^{\alpha_m t} + \int_0^t f_m(t) e^{\alpha_m (t-\tau)}\,dt.$$

By virtue principle of superposition, the solution of the boundary value problem (1.11), (1.12), (1.13) and (1.14) is giving by the series

$$z(x,t) = \sum_{m=1}^{+\infty} y_m(x) w_m(t). \tag{3.1}$$

Theorem 3.1. *Let assumption 1.1 be fulfilled. Then for any* $f \in L^2(\Omega)$ *and* $\varphi \in L_2(0,\ell)$ *which* $\dfrac{d\varphi}{dx} \in L^2(0,\ell)$, *problem*(1.20)*,* (1.21)*,* (1.22) *and* (1.23) *admits a unique solution and its represented by series* (3.1) *which converge in* E.

Proof

Consider the partial sum $S_m(x,t) = \sum_{i=1}^{m} y_i(x) w_i(t)$ of the series (3.1).
By applaying the Theorem 1.2, then it follows that

$$\left\| \sum_{i=1}^{n} y_i(x) w_i(t) \right\| \leq C \sum_{m=1}^{+\infty} \left(\int_0^T f_m^2(t)\, dt + \varphi_m^2 + \rho_m^2 \right). \tag{3.2}$$

The series $\sum_{m=1}^{+\infty} \int_0^T f_m^2(t)\, dt = \int_\Omega f^2(x,t)\, dxdt,\ \sum_{m=1}^{+\infty} \varphi_m^2$ and $\sum_{m=1}^{+\infty} \rho_m^2$ converge.
Therefore, from (3.2) it follows that the series (3.1) converge in $\mathbb{E}$.
This completes the proof of the Theorem 3.1. ⊡

Competing Interests

The authors declare that no competing interests exist.

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