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The low Dimensional Hochschild Cohomology Groups for Some Finite-dimensional Algebra

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Research Article

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Abstract

In this paper we find some results on computing the low-dimensional Hochschild cohomology groups for some finite-dimensional monomial algebra over an algebraically closed field K. The low-dimensional Hochschild cohomology groups have an important interpretations within algebra and geometry.

Keywords: Hochschild cohomology; finite dimensional algebras; low-dimensional

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1 Introduction

The main goal of this paper is to give results to use in determining the low-dimensional Hochschild cohomology groups of some finite-dimensional algebra Λ for which $\Lambda = KQ/I$. We assume throughout that Q is a quiver over an algebraically closed field K and I is an admissible ideal in KQ. Al-Kadi in ([1], Theorem 3.6) gives a general theorem on the vanishing of the second Hochschild cohomology group for most of the finite dimensional self-injective algebras of finite representation type of types D and E.

The low-dimensional Hochschild cohomology groups $\mathrm{HH}^0(\Lambda), \mathrm{HH}^1(\Lambda)$ and $\mathrm{HH}^2(\Lambda)$ (defined below) have an important interpretation within algebra such as derivations and extensions. In [2], Happel shows that $\mathrm{HH}^0(\Lambda)$ is the center of Λ and that the group $\mathrm{HH}^1(\Lambda)$ is related to derivations of an algebra. The derivations of Λ form the set $\{\delta \in \mathrm{Hom}_K(\Lambda, \Lambda) | \delta(ab) = a\delta(b) + \delta(a)b\}$. It was also noted by Gerstenhaber in [3] that there are connections to algebraic geometry. In fact, $\mathrm{HH}^2(\Lambda)$ controls the deformations of an algebra. Within algebraic geometry it is important to know whether or not $\mathrm{HH}^2(\Lambda)$ is zero. This paper is concerned with the low dimensional Hochschild cohomology groups as from an algebraic point of view and with finding the dimension of $\mathrm{HH}^i(\Lambda)$ for $i = 0, 1, 2$. Our main theorems are Theorem 3.4 and Theorem 3.6 stated as follows.

Theorem 3.4. If Q is connected and has no oriented cycles then $\dim \mathrm{Im}\, d_1 = n-1$, where n=number of vertices.

Theorem 3.6. Suppose that Q is connected and has no oriented cycles. Let $\Lambda = KQ/I$ be a

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finite-dimensional monomial algebra.Suppose also whenever $a_1 \cdots a_n$ is a minimal generator of I then dim $\mathfrak{o}(a_i)\Lambda\mathfrak{t}(a_i)$=number of arrows from $\mathfrak{o}(a_i)$to $\mathfrak{t}(a_i)$ for $i = 1, \ldots, n$.

i) If Λ has only one relation, namely $a_1 \cdots a_n$, then dim $\operatorname{Ker} d_2 =$ dim $\operatorname{Hom}(Q^1, \Lambda)$ $-\Sigma_{k=1}^{n} m_k$ where $m_k =$ dim $\mathfrak{o}(a_k)\Lambda\mathfrak{t}(a_k) - 1$.

ii) If the minimal set of generators of I is precisely the set of paths from $\mathfrak{o}(a_1)$ to $\mathfrak{t}(a_n)$. Then $\operatorname{Ker} d_2 = \operatorname{Hom}(Q^1, \Lambda)$.

iii) Special case: if dim $\mathfrak{o}(a_i)\Lambda t(a_i) = 1$ for all arrows a_i in Q then dim $\operatorname{Ker} d_2 =$ number of arrows.

Our paper is organized as follows. In Section 2, we briefly review the related definitions and theorems of Hochschild cohomology. We also include a short description of the projective resolution of [4]. In Section 3, we present the results we found to compute the dimension of the low-dimensional Hochschild cohomology groups and we conclude with an example.

2 Preliminaries

In this section we recall some standard definitions and theorems. We have not included the proofs if the results are well known or direct to prove.

Let Λ be a finite-dimensional algebra over a field K. Then any left Λ-module, say M, has a projective resolution which is an exact sequence

$$\cdots \to P_n \overset{A_n}{\to} P_{n-1} \overset{A_{n-1}}{\to} \cdots \overset{A_1}{\to} P_0 \overset{A_0}{\to} M \to 0, \tag{2.1}$$

such that each P_i is a projective module.

Notation: If

$$\cdots \to P_n \overset{A_n}{\to} P_{n-1} \overset{A_{n-1}}{\to} \cdots \overset{A_3}{\to} P_2 \overset{A_2}{\to} P_1 \overset{A_1}{\to} P_0 \overset{A_0}{\to} M \to 0,$$

is a minimal projective resolution for M then $\operatorname{Ker} A_n := \Omega^{n+1}(M)$.

Given a sequence as (2.1) we may form the complex by taking homomorphisms of each of the terms into N. This gives the complex (2.2) below:

$$0 \to \operatorname{Hom}(M, N) \overset{d_0}{\to} \operatorname{Hom}(P_0, N) \overset{d_1}{\to} \operatorname{Hom}(P_1, N) \overset{d_2}{\to} \cdots \overset{d_{n-1}}{\to} \operatorname{Hom}(P_{n-1}, N) \overset{d_n}{\to} \cdots$$

It is a sequence of modules and maps such that composition of any two adjacent maps is zero. This is the same as saying $d_n \circ d_{n-1} = 0$ that is, $\operatorname{Im} d_{n-1} \subset \operatorname{Ker} d_n$. This sequence is not necessarily exact, and leads to the extensions.

Definition 2.1. ([5], p33,p44]). Let N and M be two Λ-modules. For any projective resolution of M as in (2.1) let $d_n : \operatorname{Hom}(P_{n-1}, N) \to \operatorname{Hom}(P_n, N)$ be the induced map for all $n \geq 1$ as in (2.2). Then

$$Ext_{\Lambda}^{n}(M, N) = \operatorname{Ker} d_{n+1} / \operatorname{Im} d_n \qquad \text{for } n \geq 0,$$

where $Ext_{\Lambda}^{0}(M, N) = \operatorname{Ker} d_1$. The group $Ext_{\Lambda}^{n}(M, N)$ is called the n-th cohomology group derived from the functor $\operatorname{Hom}(-, N)$. Moreover, $Ext_{\Lambda}^{0}(M, N) = \operatorname{Hom}(M, N)$.

Theorem 2.2. *If*

$$0 \to A \overset{g}{\to} B \overset{f}{\to} C \to 0$$

is an exact sequence of vector spaces then $\dim B = \dim A + \dim C$.

Definition 2.3. Definition: ([6], p287) Let Λ be a finite-dimensional algebra over a field K. The n-th Hochschild cohomology group $\operatorname{HH}^n(\Lambda)$ is $Ext_{\Lambda^e}^{n}(\Lambda, \Lambda)$, where $\Lambda^e = \Lambda \otimes_K \Lambda^{op}$ is the enveloping algebra of Λ.

The next two theorems help us to find the zero Hochschild cohomology group:

Theorem 2.4. $\mathrm{HH}^0(\Lambda) = Z(\Lambda)$ *where* $Z(\Lambda)$ *is the center of* Λ.

Theorem 2.5. *If* Q *has no oriented cycles then* $Z(\Lambda) = K$.

To find the Hochschild cohomology groups for some finite dimensional algebras Λ, a projective resolution of Λ as Λ^e-module is needed. The next definition is written using ([4], Theorem2.9).

In general for $\Lambda = KQ/I$ where Q is a quiver and I is an admissible ideal of KQ, a minimal projective resolution of Λ as a Λ, Λ-bimodule begins:

$$\cdots \to Q^2 \stackrel{A_2}{\to} Q^1 \stackrel{A_1}{\to} Q^0 \stackrel{A_0}{\to} \Lambda \to 0,$$

where

$$Q^0 = \bigoplus_{v, vertex} \Lambda v \otimes v\Lambda,$$

$$Q^1 = \bigoplus_{a, arrow} \Lambda \mathfrak{o}(a) \otimes \mathfrak{t}(a)\Lambda,$$

$$Q^2 = \bigoplus_{x \in g^2} \Lambda \mathfrak{o}(x) \otimes \mathfrak{t}(x)\Lambda,$$

where g^2 is a minimal set of relations for the ideal I. Note that we write $\mathfrak{o}(a)$ for the origin of the arrow a and $\mathfrak{t}(a)$ for the end of a. Next we will define the maps A_0, A_1 and A_2. The map $A_0 : Q^0 \to \Lambda$, is the multiplication map so is given by $v \otimes v \mapsto v$. The map $A_1 : Q^1 \to Q^0$, is a Λ, Λ-homomorphism and is given by $\mathfrak{o}(a) \otimes \mathfrak{t}(a) \mapsto \mathfrak{o}(a) \otimes \mathfrak{o}(a)a - a\mathfrak{t}(a) \otimes \mathfrak{t}(a)$ for each arrow a. To define the map $A_2 : Q^2 \to Q^1$, let x be one of the minimal relations.

$$o(x) \otimes t(x) \mapsto \sum_{j=1}^{r} c_j (\sum_{k=1}^{s_j} a_{1j} \cdots a_{(k-1)j} \otimes a_{(k+1)j} \cdots a_{s_j j}),$$

where $a_{1j} \cdots a_{(k-1)j} \otimes a_{(k+1)j} \cdots a_{s_j j} \in \Lambda \mathfrak{o}(a_{kj}) \otimes \mathfrak{t}(a_{kj})\Lambda$.

In this paper the projective resolution is

$$0 \to Q^2 \stackrel{A_2}{\to} Q^1 \stackrel{A_1}{\to} Q^0 \stackrel{A_0}{\to} \Lambda \to 0,$$

that is, $Q^i = 0$, for $i \geq 3$. (We assume here that $Q^i = 0$, for $i \geq 3$.)

In the next section, we found some general results to describe the low dimensional Hochschild cohomology groups.

3 Results

Theorem 3.1. *Let* $\Lambda = KQ/I$. *Suppose that* Q *is connected and has no oriented cycles. Suppose that* $0 \to Q^2 \to Q^1 \to Q^0 \to \Lambda \to 0$ *is a projective resolution of* Λ. *Then* $\mathrm{HH}^0(\Lambda) \cong K$. *If* $\mathrm{Hom}(Q^2, \Lambda) \neq 0$ *and if* $\mathrm{Im}\, d_2 = 0$, *where* $d_2 : \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{Hom}(Q^2, \Lambda)$, *then we have* $\mathrm{HH}^2(\Lambda) \cong \mathrm{Hom}(Q^2, \Lambda)$ *and* $\dim \mathrm{HH}^1(\Lambda) = \dim \mathrm{HH}^0(\Lambda) - \dim \mathrm{Hom}(Q^0, \Lambda) + \dim \mathrm{Hom}(Q^1, \Lambda)$. *If* $\mathrm{Hom}(Q^2, \Lambda) = 0$, *then* $\mathrm{HH}^2(\Lambda) = 0$.

We present a summary of the proof next.

Proof. Since Q does not contain an oriented cycle then by Theorem 2.4 and Theorem 2.5 we have $\mathrm{HH}^0(\Lambda) \cong K$.

Starting with the minimal projective resolution of Λ:

$$0 \to Q^2 \overset{A_2}{\to} Q^1 \overset{A_1}{\to} Q^0 \overset{A_0}{\to} \Lambda \to 0,$$

we get the complex:

$$0 \to \mathrm{Hom}(\Lambda, \Lambda) \to \mathrm{Hom}(Q^0, \Lambda) \overset{d_1}{\to} \mathrm{Hom}(Q^1, \Lambda) \overset{d_2}{\to} \mathrm{Hom}(Q^2, \Lambda) \overset{d_3}{\to} 0.$$

We will need some assumptions on $\mathrm{Hom}(Q^2, \Lambda)$. To start consider the short exact sequence:

$$0 \to \mathrm{Ker}\, A_0 = \Omega\Lambda \to Q^0 \overset{A_0}{\to} \Lambda \to 0.$$

Then we get the following sequence:

$$0 \to \mathrm{Hom}(\Lambda, \Lambda) \to \mathrm{Hom}(Q^0, \Lambda) \to \mathrm{Hom}(\Omega\Lambda, \Lambda) \to \mathrm{HH}^1(\Lambda) \to 0, \tag{3.1}$$

where $\mathrm{HH}^1(\Lambda) = Ext^1_{\Lambda^e}(\Lambda, \Lambda)$.

By repeating the steps but with a short exact sequence containing $\Omega\Lambda$. i.e. by using the short exact sequence:

$$0 \to \mathrm{Ker}\, A_1 = \Omega^2\Lambda \to Q^1 \overset{A_1}{\to} \Omega\Lambda \to 0,$$

we get the following sequence:

$$0 \to \mathrm{Hom}(\Omega\Lambda, \Lambda) \to \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{Hom}(\Omega^2\Lambda, \Lambda) \to \mathrm{HH}^2(\Lambda) \to 0, \tag{3.2}$$

where $\mathrm{HH}^2(\Lambda) = Ext^1_{\Lambda^e}(\Omega\Lambda, \Lambda)$.

We also have the short exact sequence that contains $\Omega^2\Lambda$:

$$0 \to \mathrm{Ker}\, A_2 = \Omega^3\Lambda \to Q^2 \overset{A_2}{\to} \Omega^2\Lambda \to 0.$$

But $\mathrm{Ker}\, A_2 = \Omega^3\Lambda = 0$, so $Q^2 \cong \Omega^2\Lambda$. Now substitute it in (3.2) to get:

$$0 \to \mathrm{Hom}(\Omega\Lambda, \Lambda) \to \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{Hom}(Q^2, \Lambda) \to \mathrm{HH}^2(\Lambda) \to 0. \tag{3.2a}$$

If we make the assumption that $\mathrm{Hom}(Q^2, \Lambda) = 0$ then it follows directly from equation (3.2a) that $\mathrm{Hom}(\Omega\Lambda, \Lambda) \cong \mathrm{Hom}(Q^1, \Lambda)$ and $\mathrm{HH}^2(\Lambda) = 0$.

Now if we assume $\mathrm{Hom}(Q^2, \Lambda) \neq 0$ and $\mathrm{Im}\, d_2 = 0$, then $\mathrm{HH}^2(\Lambda) = \mathrm{Ker}\, d_3/\mathrm{Im}\, d_2 \cong \mathrm{Hom}(Q^2, \Lambda)$. Again it follows directly from (3.2a) that $\mathrm{Hom}(\Omega\Lambda, \Lambda) \cong \mathrm{Hom}(Q^1, \Lambda)$.

Now in sequence (3.1), we know that $\mathrm{Hom}(\Lambda, \Lambda) \cong Z(\Lambda) = \mathrm{HH}^0(\Lambda)$ and that $\mathrm{Hom}(\Omega\Lambda, \Lambda) \cong \mathrm{Hom}(Q^1, \Lambda)$, so we get:

$$0 \to \mathrm{HH}^0(\Lambda) \to \mathrm{Hom}(Q^0, \Lambda) \to \mathrm{Hom}(Q^1, \Lambda) \to \mathrm{HH}^1(\Lambda) \to 0. \tag{3.1a}$$

So $\dim\, \mathrm{HH}^0(\Lambda) - \dim\, \mathrm{Hom}(Q^0, \Lambda) + \dim\, \mathrm{Hom}(Q^1, \Lambda) - \dim\, \mathrm{HH}^1(\Lambda) = 0$. Therefore, $\dim\, \mathrm{HH}^1(\Lambda) = \dim\, \mathrm{HH}^0(\Lambda) - \dim\, \mathrm{Hom}(Q^0, \Lambda) + \dim\, \mathrm{Hom}(Q^1, \Lambda)$.

□

The next results describe $\mathrm{Hom}(Q^i, \Lambda)$, for $i = 0, 1, 2$.

Theorem 3.2. *There is an isomorphism of vector spaces* $\mathrm{Hom}(\Lambda e \otimes f\Lambda, \Lambda) \cong e\Lambda f$.

Proof. Let $\alpha : \mathrm{Hom}(\Lambda e \otimes f\Lambda, \Lambda) \to e\Lambda f$ be defined by $\phi \mapsto \phi(e \otimes f)$, where $\phi : \Lambda e \otimes f\Lambda \to \Lambda$. Then it is direct to show that α is an isomorphism. □

Theorem 3.3. *With the notation of this section and section 1,*

i) $\mathrm{Hom}(Q^0, \Lambda) = \bigoplus_{v, vertex} \mathfrak{o}(v)\Lambda\mathfrak{t}(v)$.

ii) $\mathrm{Hom}(Q^1, \Lambda) = \bigoplus_{a, arrow} \mathfrak{o}(a)\Lambda\mathfrak{t}(a)$.

iii) $\mathrm{Hom}(Q^2, \Lambda) = \bigoplus_{x \in g^2} \mathfrak{o}(x)\Lambda\mathfrak{t}(x)$.

Proof. i) $\mathrm{Hom}(Q^0,\Lambda) = \mathrm{Hom}(\bigoplus_{v,vertex}\Lambda\mathfrak{o}(v)\otimes\mathfrak{t}(v)\Lambda,\Lambda) = \bigoplus_{v,vertex}\mathrm{Hom}(\Lambda\mathfrak{o}(v)\otimes\mathfrak{t}(v)\Lambda,\Lambda) \cong \bigoplus_{v,vertex}\mathfrak{o}(v)\Lambda\mathfrak{t}(v)$ by Theorem 3.2. Similarly, we can prove ii) and iii). □

Remarks: i) $\dim\ \mathrm{Hom}(Q^0,\Lambda) = \sum_{v,vertex}\dim\ \mathfrak{o}(v)\Lambda\mathfrak{t}(v)$.
ii) $\dim\ \mathrm{Hom}(Q^1,\Lambda) = \sum_{a,arrorw}\dim\ \mathfrak{o}(a)\Lambda\mathfrak{t}(a)$.
iii) $\dim\ \mathrm{Hom}(Q^2,\Lambda) = \sum_{x\in g^2}\dim\ \mathfrak{o}(x)\Lambda\mathfrak{t}(x)$.

Theorem 3.4. *If Q is connected and has no oriented cycles then $\dim\ \mathrm{Im}\, d_1 = n-1$, where n=number of vertices.*

Proof. Since $d_1 : \mathrm{Hom}(Q^0,\Lambda) \to \mathrm{Hom}(Q^1,\Lambda)$, then we get the exact sequence:

$$0 \to \mathrm{Ker}\, d_1 \to \mathrm{Hom}(Q^0,\Lambda) \to \mathrm{Im}\, d_1 \to 0.$$

Then by Theorem 2.2 we have $\dim\ \mathrm{Im}\, d_1 = \dim\ \mathrm{Hom}(Q^0,\Lambda) - \dim\ \mathrm{Ker}\, d_1$, and $\mathrm{Hom}(Q^0,\Lambda) \cong \bigoplus_{v,vertix}\mathfrak{o}(v)\Lambda\mathfrak{t}(v)$. So $\dim\ \mathrm{Hom}(Q^0,\Lambda) = n$, since Q has no oriented cycles. Also $\mathrm{HH}^0(\Lambda) \cong K$. Therefore $\dim\ \mathrm{HH}^0(\Lambda) = 1$. On the other hand, $\mathrm{HH}^0(\Lambda) = Ext^0(\Lambda,\Lambda) = \mathrm{Ker}\, d_1$ by Definition 2.1. Hence, $\dim\ \mathrm{Ker}\, d_1 = 1$. Therefore, $\dim\ \mathrm{Im}\, d_1 = n-1$. □

We know that $\mathrm{HH}^1(\Lambda) = \mathrm{Ker}\, d_2/\mathrm{Im}\, d_1$. By using Theorem 3.4 we can find $\dim\ \mathrm{Im}\, d_1$. To find $\dim\ \mathrm{Ker}\, d_2$, Theorem 3.6 below has been identified. A definition of a monomial algebra is needed first.

Definition 3.5. ([1], Definition 1.17) Let $\Lambda = KQ/I$. Then Λ is a monomial algebra if I is generated by a set of paths in KQ each of length at least 2.

Theorem 3.6. *Suppose that Q is connected and ha s no oriented cycles. Let $\Lambda = KQ/I$ be a finite-dimensional monomial algebra.Suppose also whenever $a_1\cdots a_n$ is a minimal generator of I then $\dim\ \mathfrak{o}(a_i)\Lambda\mathfrak{t}(a_i)$=number of arrows from $\mathfrak{o}(a_i)$to $\mathfrak{t}(a_i)$ for $i = 1,\ldots,n$.*

i) If Λ has only one relation, namely $a_1\cdots a_n$, then $\dim\ \mathrm{Ker}\, d_2 = \dim\ \mathrm{Hom}(Q^1,\Lambda) - \Sigma_{k=1}^{n} m_k$ where $m_k = \dim\ \mathfrak{o}(a_k)\Lambda\mathfrak{t}(a_k) - 1$.

ii) If the minimal set of generators of I is precisely the set of paths from $\mathfrak{o}(a_1)$ to $\mathfrak{t}(a_n)$. Then $\mathrm{Ker}\, d_2 = \mathrm{Hom}(Q^1,\Lambda)$.

iii) Special case: if $\dim\ \mathfrak{o}(a_i)\Lambda t(a_i) = 1$ for all arrows a_i in Q then $\dim\ \mathrm{Ker}\, d_2$ = number of arrows.

Proof. Since we have the map $d_1 : \mathrm{Hom}(Q^0,\Lambda) \to \mathrm{Hom}(Q^1,\Lambda)$, then we get the exact sequence:

$$0 \to \mathrm{Ker}\, d_2 \to \mathrm{Hom}(Q^1,\Lambda) \to \mathrm{Im}\, d_2 \to 0.$$

Therefore,
$\dim\ \mathrm{Ker}\, d_2 = \ dim\,\mathrm{Hom}(Q^1,\Lambda) - \dim\ \mathrm{Im}\, d_2$ and $\mathrm{Hom}(Q^1,\Lambda) \cong \bigoplus_{(a,arrow)}\mathfrak{o}(a)\Lambda\mathfrak{t}(a)$. Since Λ is a monomial algebra, I is generated by monomial relations. Fix a minimal generation set of monomials for I. Suppose that $r = a_1\cdots a_n$ is one of these minimal relations. Then a typical element of $\mathfrak{o}(a_i)\Lambda\mathfrak{t}(a_i)$ is a linear combination of paths from $\mathfrak{o}(a_i)$ to $\mathfrak{t}(a_i)$. By hypothesis, a path from $\mathfrak{o}(a_i)$ to $\mathfrak{t}(a_i)$ is an arrow from $\mathfrak{o}(a_i)$ to $\mathfrak{t}(a_i)$. So $\mathfrak{o}(a_i)\Lambda\mathfrak{t}(a_i)$ has typical element of the form $c_{a_i}a_i + \Sigma_{j=1}^{m_k} c_{i_j}\beta_{i_j}$, for some $c_{a_i}, c_{i_j} \in K$ and arrows β_{i_j} from $\mathfrak{o}(a_i)$ to $\mathfrak{t}(a_i)$ $(\beta_{i_j} \neq a_i)$. Now let $g \in \mathrm{Hom}(Q^1,\Lambda)$. Then $g : Q^1 \to \Lambda$ is given by $\mathfrak{o}(a)\otimes\mathfrak{t}(a) \mapsto \mathfrak{o}(a)\lambda_a\mathfrak{t}(a)$ for each arrow a. Also

$$\begin{aligned} gA_2(\mathfrak{o}(r)\otimes\mathfrak{t}(r)) &= g(\mathfrak{o}(a_1)\otimes a_2\cdots a_n + a_1\otimes a_3\cdots a_n + \ldots + a_1\cdots a_{n-1}\otimes\mathfrak{t}(a_n)) \\ &= g(\mathfrak{o}(a_1)\otimes\mathfrak{t}(a_1))a_2\cdots a_n + a_1 g(\mathfrak{o}(a_2)\otimes\mathfrak{t}(a_2))a_3\cdots a_n + \ldots \\ &\quad + a_1\cdots a_{n-1}g(\mathfrak{o}(a_n)\otimes\mathfrak{t}(a_n)) \\ &= (\mathfrak{o}(a_1)\lambda_{a_1}\mathfrak{t}(a_1))a_2\cdots a_n + a_1(\mathfrak{o}(a_2)\lambda_{a_2}\mathfrak{t}(a_2))a_3\cdots a_n + \ldots \end{aligned}$$

$+a_1 \cdots a_{n-1}(\mathfrak{o}(a_n)\lambda_{a_n}\mathfrak{t}(a_n))$.

So if $\lambda_{a_i} = c_{a_i}a_i + \Sigma_{j=1}^{m_k} c_{i_j}\beta_{i_j}$ then $gA_2(\mathfrak{o}(r)\otimes\mathfrak{t}(r)) = (c_{a_1}a_1 + \Sigma_{j=1}^{m_1}c_{1_j}\beta_{1_j})a_2\cdots a_n + a_1(c_{a_2}a_2 + \Sigma_{j=1}^{m_2}c_{2_j}\beta_{2_j})a_3\cdots a_n + \ldots + a_1\cdots a_{n-1}(c_{a_n}a_n + \Sigma_{j=1}^{m_n}c_{n_j}\beta_{n_j})$. (3.3)

For i) assume that Λ has only one relation, say $r = a_1\cdots a_n$. Since $\beta_{i_j} \neq a_i$, so $\beta_{1_j}a_2\cdots a_n \neq 0, a_1\beta_{2_j}a_3\cdots a_n \neq 0$, etc. Moreover, they are all linearly independent. Let $g \in \operatorname{Ker} d_2$, then $gA_2 = 0$ and $g \in \operatorname{Hom}(Q^1,\Lambda)$. Hence from (3.3), $c_{i_j} = 0$, for all i. Therefore, $g(\mathfrak{o}(a_i)\otimes\mathfrak{t}(a_i)) = c_{a_i}a_i$ for $i = 1,\ldots,n$ and $g(\mathfrak{o}(a)\otimes\mathfrak{t}(a)) = \mathfrak{o}(a)\lambda\mathfrak{t}(a)$ for $a \neq a_1,\ldots,a_n$. Hence $\dim \operatorname{Ker} d_2 = \dim \operatorname{Hom}(Q^1,\Lambda) - \Sigma_{k=1}^{n} m_k$, where $m_k = \dim\ \mathfrak{o}(a_k)\Lambda\mathfrak{t}(a_k) - 1$.

For ii) suppose each minimal generator of I is of the form $r = \gamma_1\cdots\gamma_n$, where γ_i is some a_i or β_{i_j}. Recall that β_{i_j} is an arrow from $\mathfrak{o}(a_i)$ to $\mathfrak{t}(a_i)$. By using similar process to the one used in i) we get for $g \in \operatorname{Hom}(Q^1,\Lambda)$ that $gA_2(\mathfrak{o}(r)\otimes\mathfrak{t}(r)) = (\mathfrak{o}(\gamma_1)\lambda_{\gamma_1}\mathfrak{t}(\gamma_1))\gamma_2\cdots\gamma_n + \gamma_1(\mathfrak{o}(\gamma_2)\lambda_{\gamma_2}\mathfrak{t}(\gamma_2))\gamma_3\cdots\gamma_n + \ldots + \gamma_1\cdots\gamma_{n-1}(\mathfrak{o}(\gamma_n)\lambda_{\gamma_n}\mathfrak{t}(\gamma_n))$. Since $\lambda_{\gamma_i} \in \mathfrak{o}(\gamma_i)\Lambda\mathfrak{t}(\gamma_i) = \mathfrak{o}(a_i)\Lambda\mathfrak{t}(a_i)$ we may write $\lambda_{\gamma_i} = c_{\gamma_i}\gamma_i + \Sigma_{j=1}^{m_k}c_{i_j}\beta_{i_j}$, for some $c_{\gamma_i}, c_{i_j} \in K$. Then as in equation (3.3) we have $gA_2(\mathfrak{o}(r)\otimes\mathfrak{t}(r)) = (c_{\gamma_1}\gamma_1 + \Sigma_{j=1}^{m_1}c_{1_j}\beta_{1_j})\gamma_2\cdots\gamma_n + \gamma_1(c_{\gamma_2}\gamma_2 + \Sigma_{j=1}^{m_2}c_{2_j}\beta_{2_j})\gamma_3\cdots\gamma_n + \ldots + \gamma_1\cdots\gamma_{n-1}(c_{\gamma_n}\gamma_n + \Sigma_{j=1}^{m_n}c_{n_j}\beta_{n_j}) = 0$. Therefore, $g \in \operatorname{Ker} d_2$ so $\operatorname{Ker} d_2 = \operatorname{Hom}(Q^1,\Lambda)$.

iii) Special case: assume $\dim\ \mathfrak{o}(a_i)\Lambda\mathfrak{t}(a_i) = 1$ for all arrows $a_i \in Q$. Then $\mathfrak{o}(a_i)\lambda_{a_i}\mathfrak{t}(a_i) = c_{a_i}a_i$, where $c_{a_i} \in K$. So, for $g \in \operatorname{Hom}(Q^1,\Lambda)$ and any relation $r = a_1\cdots a_n$, the equation (3.3) becomes

$$gA_2(\mathfrak{o}(r)\otimes\mathfrak{t}(r)) = c_{a_1}a_1\cdots a_n + a_1c_{a_2}a_2\cdots a_n + \ldots + a_1\cdots a_{n-1}c_{a_n}a_n$$

$$= (c_{a_1} + c_{a_2} + \ldots + c_{a_n})(a_1\cdots a_n) = 0.$$

Therefore, $g \in \operatorname{Ker} d_2$ so $\operatorname{Ker} d_2 = \operatorname{Hom}(Q^1,\Lambda)$. Since $\dim\ \mathfrak{o}(a_i)\Lambda\mathfrak{t}(a_i) = 1$, then $\dim\operatorname{Hom}(Q^1,\Lambda)$ = number of arrows. Hence $\dim\ \operatorname{Ker} d_2$ = number of arrows. □

Note that once we have described $\operatorname{Ker} d_2$, then we can find $\operatorname{Im} d_2$. Thus we can describe $\operatorname{HH}^1(\Lambda)$ and $\operatorname{HH}^2(\Lambda)$, in the cases $Q^i = 0\ \forall i \geq 3$, i.e., where $\operatorname{Ker} d_3 = \operatorname{Hom}(Q^2,\Lambda)$.

An Example. Let $\Lambda = KQ/I$ where Q is the quiver with two arrows α and β from the vertex 1 to the vertex 2, an arrow γ from the vertex 2 to the vertex 3 and $I = \langle\alpha\gamma\rangle$. The algebra $\mathcal{Q}$ is connected and has no oriented cycles and Λ has only one relation. From Theorem 3.6(i), $\dim\operatorname{Ker} d_2 = \operatorname{Hom}(Q^1,\Lambda) - (m_1 + m_2)$, where $m_1 =$ (number of arrows from $\mathfrak{o}(\alpha)$ to $\mathfrak{t}(\alpha)) - 1$, so $m_1 = 1$, and $m_2 =$ (number of arrows from $\mathfrak{o}(\gamma)$ to $\mathfrak{t}(\gamma)) - 1$, so $m_2 = 0$. By using Theorem 3.3, $\dim\operatorname{Hom}(Q^1,\Lambda) = \Sigma_{a,arrow}\dim\ \mathfrak{o}(a)\Lambda\mathfrak{t}(a) = \dim\ e_1\Lambda e_2 + \dim\ e_1\Lambda e_2 + \dim\ e_2\Lambda e_3 = 2 + 2 + 1 = 5$. Hence, $\dim\ \operatorname{Ker} d_2 = 5 - 1 = 4$. Therefore, $\dim\ \operatorname{HH}^1(\Lambda) = \dim\ \operatorname{Ker} d_2 - \dim\ \operatorname{Im} d_1 = 4 - 2 = 2$, since $\dim\ \operatorname{Im} d_1 = n - 1 = 3 - 1 = 2$ from Theorem 3.4.

Now we will find $\operatorname{HH}^2(\Lambda)$. Since $d_3 : \operatorname{Hom}(Q^2,\Lambda) \to 0$, then $\operatorname{Ker} d_3 = \operatorname{Hom}(Q^2,\Lambda)$. Again by using Theorem 3.3, $\dim\ \operatorname{Hom}(Q^2,\Lambda) = \Sigma_{r\in g^2}\dim\ \mathfrak{o}(r)\Lambda\mathfrak{t}(r) = \dim\ e_1\Lambda e_3 = 1$, since $g^2 = \{\alpha\gamma\}$. On the other hand, $\dim\ \operatorname{Im} d_2 = m_1 + m_2 = 1$, since $\dim\ \operatorname{Ker} d_2 = \dim\ \operatorname{Hom}(Q^1,\Lambda) - \dim\ \operatorname{Im} d_2$. Hence, $\dim\ \operatorname{HH}^2(\Lambda) = \dim\ \operatorname{Ker} d_3 - \dim\ \operatorname{Im} d_2 = 1 - 1 = 0$.

4 Conclusion

We have introduced in Section 3 some results to help in computing the low-dimensional Hochschild cohomology groups for some finite-dimensional monomial algebra Λ over an algebraically closed field K.

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Competing Interests

The author declares that no competing interests exist.

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