British Journal of Applied Science & Technology
10(6): 1-19, 2015, Article no.BJAST.19138
ISSN: 2231-0843

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Global Stability of Almost Periodic Solution of a Discrete Multispecies Gilpin-Ayala Mutualism System

Hui Zhang[1*]

[1] *Mathematics and OR Section, Xi'an Research Institute of High-tech Hongqing Town, Xi'an, Shaanxi 710025, China.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJMCS/2015/19138
Editor(s):
(1) Qing-Wen Wang, Department of Mathematics, Shanghai University, P.R. China.
Reviewers:
(1) Rachana Pathak, University of Lucknow, India.
(2) Anonymous, University of Technology, China.
Complete Peer review History: http://sciencedomain.org/review-history/10158

Original Research Article

Received: 27th May 2015
Accepted: 16th June 2015
Published: 14th July 2015

Abstract

This paper discusses a discrete multispecies Gilpin-Ayala mutualism system. We first study the permanence and global attractivity of the system. Assume that the coefficients in the system are almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is globally attractive and uniformly asymptotically stable by constructing a suitable Liapunov function, respectively. Two examples together with numerical simulation indicate the feasibility of the main results.

Keywords: Almost periodic solution; discrete; gilpin-ayala mutualism system; permanence; global attractivity; uniformly asymptotically stable.

2010 Mathematics Subject Classification: 39A11

**Corresponding author: E-mail: zh53054958@163.com*

1 Introduction

The mutualism system[1] has been studied by more and more scholars. Topics such as permanence, global attractivity and global stability of continuous differential mutualism system were extensively investigated(see[2,3,4,5] and the references cited therein). Xia et al.[2] studied a Lotka-Volterra type mutualism system with several delays. Some new and interesting sufficient conditions are obtained for the global existence of positive periodic solutions of the mutualism system. Their method is based on Mawhin's coincidence degree and novel estimation techniques for the a priori bounds of unknown solutions. In addition, some recent attention was on the permanence and global stability of discrete mutualism system, and many excellent results have been derived(see[6,7,8,9,10] and the references cited therein).

For the last decades, as far as the continuous and discrete multispecies Lotka-Volterra ecosystem is concerned(see[8,11,12,13,14,15,16] and the references cited therein). However, the Lotka-Volterra type models have often been severely criticized. One of the criticisms is that in such a model, the per capita rate of change of the density of each species is a linear function of densities of the interacting species. In 1973, Gilpin, Ayala et al.[17,18] claimed that more complicated competition system are needed to study qualitative properties of the systems. To this aim, they proposed several competition models. One of the models is the following competition system

$$\dot{N}_i(t) = r_i N_i\left(1 - (\frac{N_i}{K_i})^{\theta_i} - \sum_{j=1, j\neq i}^{n} \alpha_{ij}\frac{N_j}{K_j}\right).$$

Fan and Wang[19] further proposed delay Gilpin-Ayala type competition model

$$\dot{y}_i(t) = y_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij} y_j^{\theta_{ij}}(t-\tau_{ij}(t))\right],$$

$$i = 1, 2, \cdots, n.$$

By applying the coincidence degree theory, they obtained a set of easily verifiable sufficient conditions for the existence of at least one positive periodic solution of the model. Chen et al.[20] had investigated the dynamic behavior of the following discrete n-species Gilpin-Ayala competition model

$$x_i(k+1) = x_i(k)\exp\left\{b_i(k) - \sum_{j=1}^{n} a_{ij}(k)(x_j(k))^{\theta_{ij}}\right\}.$$

For general nonautonomous case, sufficient conditions which ensure the permanence and the global stability of the system are obtained; For periodic case, sufficient conditions which ensure the existence of an unique globally stable positive periodic solution of the system are obtained. Li and Chen[21] obtained that r of the species in the above system are permanent and stabilize at a unique strictly positive almost periodic solution of the corresponding subsystem, which is globally attractive, while the remaining $n-r$ species are driven to extinction.

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, harvesting. So it is usual to assume the periodicity of parameters in the systems. However, if the various constituent components of the temporally nonuniform environment is with incommensurable (non-integral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. For this reason, the assumption of almost periodicity is more realistic, more important and more general when we consider the effects of the environmental factors. In fact, there have been many nice works on the positive almost periodic solutions of continuous and discrete dynamics model with almost periodic coefficients(see[5,7,8,9,10,22,23,24,25, 26,27,28] and the references cited therein). Liao and Zhang[7] studied a discrete mutualism model with variable delays. By means of an almost periodic functional hull theory, sufficient conditions are established for the existence and uniqueness of globally attractive almost periodic solution to the system.

Motivated by above, in this paper, we are concerned with the following discrete multispecies Gilpin-Ayala mutualism system.

$$x_i(k+1) = x_i(k)\exp\left\{a_i(k) - b_i(k)(x_i(k))^{\theta_{ii}} + \sum_{j=1,j\neq i}^{n} c_{ij}(k)\frac{(x_j(k))^{\theta_{ij}}}{d_{ij}(k)+(x_j(k))^{\theta_{ij}}}\right\}, \quad (1.1)$$

where $i = 1, 2, \cdots, n$; $x_i(k)$ stand for the densities of species x_i at the kth generation, $a_i(k)$ represent the natural growth rates of species x_i at the kth generation, $b_i(k)$ are the intraspecific effects of the kth generation of species x_i on own population, and $c_{ij}(k)$ measure the interspecific mutualism effects of the kth generation of species x_j on species $x_i (i, j = 1, 2, \cdots, n, i \neq j)$, $d_{ij}(k)$ are positive control sequences. θ_{ii} and θ_{ij} are positive constants.

Denote as Z and Z^+ the set of integers and the set of nonnegative integers, respectively. For any bounded sequence $\{g(n)\}$ defined on Z, define $g^u = \sup_{n\in Z} g(n), g^l = \inf_{n\in Z} g(n)$.

Throughout this paper, we assume that:

(H1) $\{a_i(k)\}, \{b_i(k)\}, \{c_{ij}(k)\}$ and $\{d_{ij}(k)\}$ are bounded nonnegative almost periodic sequences such that

$$0 < a_i^l \leq a_i(k) \leq a_i^u, \quad 0 < b_i^l \leq b_i(k) \leq b_i^u,$$
$$0 < c_{ij}^l \leq c_{ij}(k) \leq c_{ij}^u, \quad 0 < d_{ij}^l \leq d_{ij}(k) \leq d_{ij}^u.$$

From the point of view of biology, in the sequel, we assume that $\mathbf{x}(0) = (x_1(0), x_2(0), \cdots, x_n(0)) > \mathbf{0}$. Then it is easy to see that, for given $\mathbf{x}(0) > \mathbf{0}$, the system (1.1) has a positive sequence solution $\mathbf{x}(k) = (x_1(k), x_2(k), \cdots, x_n(k))(k \in Z^+)$ passing through $\mathbf{x}(0)$.

To the best of our knowledge, this is the first paper to investigate the global stability of positive almost periodic solution of discrete multispecies Gilpin-Ayala mutualism system. The aim of this paper is to obtain sufficient conditions for the existence of a unique globally attractive and uniformly asymptotically stable almost periodic solution of system (1.1) with condition (H1), by utilizing the theory of difference equation and constructing a suitable Lyapunov function and applying the analysis technique of papers [9,10,20,22,29].

The remaining part of this paper is organized as follows: In Section 2, we will introduce some definitions and several useful lemmas. In Section 3, by applying the theory of difference inequality, we present the permanence results for system (1.1). In Section 4, we establish the sufficient conditions for the existence of a unique globally attractive and uniformly asymptotically stable almost periodic solution of system (1.1). The main results are illustrated by two examples with numerical simulation in Section 5. Finally, the conclusion ends with brief remarks in the last section.

2 Preliminaries

Firstly, we give the definitions of the terminologies involved.

Definition 2.1([30]) A sequence $x : Z \to R$ is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z : | x(n+\tau) - x(n) | < \varepsilon, \forall n \in Z\}$$

is a relatively dense set in Z for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$| x(n+\tau) - x(n) | < \varepsilon, \quad \forall n \in Z.$$

τ is called an ε-translation number of $x(n)$.

Definition 2.2([31]) A sequence $x : Z^+ \to R$ is called an asymptotically almost periodic sequence if

$$x(n) = p(n) + q(n), \ \ \forall n \in Z^+,$$

where $p(n)$ is an almost periodic sequence and $\lim_{n\to\infty} q(n) = 0$.

Definition 2.3([32]) A solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) is said to be globally attractive if for any other solution $(x_1^*(k), x_2^*(k), \cdots, x_n^*(k))$ of system (1.1), we have

$$\lim_{k\to+\infty}(x_i^*(k) - x_i(k)) = 0, \ i = 1, 2, \cdots, n.$$

Now, we present some results which will play an important role in the proof of the main results.

Lemma 2.1([33]) If $\{x(n)\}$ is an almost periodic sequence, then $\{x(n)\}$ is bounded.

Lemma 2.2([34]) $\{x(n)\}$ is an almost periodic sequence if and only if, for any sequence $m_i \subset Z$, there exists a subsequence $\{m_{i_k}\} \subset \{m_i\}$ such that the sequence $\{x(n + m_{i_k})\}$ converges uniformly for all $n \in Z$ as $k \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 2.3([31]) $\{x(n)\}$ is an asymptotically almost periodic sequence if and only if, for any sequence $m_i \subset Z$ satisfying $m_i > 0$ and $m_i \to \infty$ as $i \to \infty$ there exists a subsequence $\{m_{i_k}\} \subset \{m_i\}$ such that the sequence $\{x(n + m_{i_k})\}$ converges uniformly for all $n \in Z^+$ as $k \to \infty$.

Lemma 2.4([33]) Suppose that $\{p_1(n)\}$ and $\{p_2(n)\}$ are almost periodic real sequences. Then $\{p_1(n) + p_2(n)\}$ and $\{p_1(n)p_2(n)\}$ are almost periodic; $\frac{1}{p_1(n)}$ is also almost periodic provided that $p_1(n) \neq 0$ for all $n \in Z$. Moreover, if $\varepsilon > 0$ is an arbitrary real number, then there exists a relatively dense set that is $\varepsilon-$almost periodic common to $\{p_1(n)\}$ and $\{p_2(n)\}$.

Lemma 2.5([20]) Assume that sequence $\{x(n)\}$ satisfies $x(n) > 0$ and

$$x(n+1) \leq x(n) \exp\{a(n) - b(n)x^{\alpha}(n)\} \quad (2.1)$$

for $n \in N$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants, α is a positive constant. Then

$$\limsup_{n\to+\infty} x(n) \leq \left(\frac{1}{\alpha b^l}\right)^{\frac{1}{\alpha}} \exp\{a^u - \frac{1}{\alpha}\}. \quad (2.2)$$

Lemma 2.6([20]) Assume that sequence $\{x(n)\}$ satisfies

$$x(n+1) \geq x(n) \exp\{a(n) - b(n)x^{\alpha}(n)\}, \quad n \geq N_0,$$

$\limsup_{n\to+\infty} x(k) \leq x^*$ and $x(N_0) > 0$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants, α is a positive constant and $N_0 \in N$. Then

$$\liminf_{n\to+\infty} x(n) \geq \left(\frac{a^l}{b^u}\right)^{\frac{1}{\alpha}} \exp\{a^l - b^u(x^*)^{\alpha}\}. \quad (2.3)$$

Consider the following almost periodic difference system:

$$x(n+1) = f(n, x(n)), \quad n \in Z^+, \quad (2.4)$$

where $f : Z^+ \times S_B \to R^k, S_B = \{x \in R^k : \parallel x \parallel < B\}$, and $f(n,x)$ is almost periodic in n uniformly for $x \in S_B$ and is continuous in x. The product system of (2.1) is the following system:

$$x(n+1) = f(n, x(n)), \quad y(n+1) = f(n, y(n)), \quad (2.5)$$

and Zhang[34,35] obtained the following Theorem.

Theorem 2.7([34,35]) Suppose that there exists a Lyapunov function $V(n,x,y)$ defined for $n \in Z^+$, $\parallel x \parallel < B$, $\parallel y \parallel < B$ satisfying the following conditions:

(i) $a(\parallel x - y \parallel) \leq V(n,x,y) \leq b(\parallel x - y \parallel)$, where $a, b \in K$ with $K = \{a \in C(R^+, R^+) : a(0) = 0$ and a is increasing$\}$;

(ii) $\parallel V(n, x_1, y_1) - V(n, x_2, y_2) \parallel \leq L[\parallel x_1 - x_2 \parallel + \parallel y_1 - y_2 \parallel]$, where $L > 0$ is a constant;

(iii) $\Delta V_{(2.5)}(n,x,y) \leq -\alpha V(n,x,y)$, where $0 < \alpha < 1$ is a constant, and

$$\Delta V_{(2.5)}(n,x,y) \equiv V(n+1, f(n,x),$$

$$f(n,y)) - V(n,x,y).$$

Moreover, if there exists a solution $\varphi(n)$ of (2.4) such that $\parallel \varphi(n) \parallel \leq B^* < B$ for $n \in Z^+$, then there exists a unique uniformly asymptotically stable almost periodic solution $p(n)$ of system (2.4) which is bounded by B^*. In particular, if $f(n,x)$ is periodic of period ω, then there exists a unique uniformly asymptotically stable periodic solution of system (2.4) of period ω.

3 Permanence

In this section, we establish a permanence result for system (1.1), which can be found by Lemma 2.5 and 2.6.

Proposition 3.1 Assume that (H1) holds. Then any positive solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) satisfies

$$m_i \leq \liminf_{k\to+\infty} x_i(k) \leq \limsup_{k\to+\infty} x_i(k) \leq M_i, \quad (3.1)$$

where

$$M_i = \left(\frac{1}{\theta_{ii} b_i^l}\right)^{\dfrac{1}{\theta_{ii}}} \exp\left\{a_i^u + \sum_{j=1,j\neq i}^{n} c_{ij}^u - \frac{1}{\theta_{ii}}\right\},$$

$$m_i = \left(\frac{a^l}{b^u}\right)^{\dfrac{1}{\alpha}} \exp\{a^l - b^u (M_i)^\alpha\},$$

$i = 1, 2, \cdots, n.$

Proof. From the equations of system (1.1), we have

$$x_i(k) \exp\left\{a_i(k) - b_i(k)(x_i(k))^{\theta_{ii}}\right\} \le x_i(k+1) \le x_i(k) \exp$$

$$\left\{a_i^u + \textstyle\sum_{j=1,j\neq i}^n c_{ij}^u - b_i(k)(x_i(k))^{\theta_{ii}}\right\}.$$

As the direct conclusion of Lemma 2.5 and 2.6, the inequality (3.1) is completed.

Theorem 3.2 Assume that (H1) holds, then system (1.1) is permanent.

The next result tells us that there exist solutions of system (1.1) totally in the interval of Theorem 3.2. We denote by Ω the set of all solutions $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) satisfying $m_i \le x_i(k) \le M_i (i = 1, 2, \cdots, n)$ for all $k \in Z^+$.

Proposition 3.3 Assume that (H1) holds. Then $\Omega \neq \Phi$.

Proof. By the almost periodicity of $\{a_i(k)\}, \{b_i(k)\}, \{c_{ij}(k)\}$ and $\{d_{ij}(k)\}$, there exists an integer valued sequence $\{\delta_p\}$ with $\delta_p \to \infty$ as $p \to \infty$ such that

$$a_i(k+\delta_p) \to a_i(k),\ \ b_i(k+\delta_p) \to b_i(k), c_{ij}(k+\delta_p) \to c_{ij}(k),$$

$$d_{ij}(k+\delta_p) \to d_{ij}(k), \ \text{ as } \ p \to +\infty.$$

Let ε be an arbitrary small positive number. It follows from Theorem 3.1 that there exists a positive integer N_0 such that

$$m_i - \varepsilon \le x_i(k) \le M_i + \varepsilon, \ \ k > N_0.$$

Write $x_{ip}(k) = x_i(k + \delta_p)$ for $k \ge N_0 - \delta_p$ and $p = 1, 2, \cdots$. For any positive integer q, it is easy to see that there exists a sequence $\{x_{ip}(k) : p \ge q\}$ such that the sequence $\{x_{ip}(k)\}$ has a subsequence, denoted by $\{x_{ip}(k)\}$ again, converging on any finite interval of Z as $p \to \infty$. Thus we have a sequence $\{y_i(k)\}$ such that

$x_{ip}(k) \to y_i(k)$ for $k \in Z$ as $p \to +\infty$.

This, combining with gives us

$$x_i(k+1+\delta_p) = x_i(k+\delta_p) \exp\left\{a_i(k+\delta_p) - b_i(k+\delta_p)(x_i(k+\delta_p))^{\theta_{ii}} + \sum_{j=1,j\neq i}^{n} c_{ij}(k+\delta_p) \frac{(x_j(k+\delta_p))^{\theta_{ij}}}{d_{ij}(k+\delta_p) + (x_j(k+\delta_p))^{\theta_{ij}}}\right\}, i = 1, 2, \cdots, n$$

$$y_i(k+1) = y_i(k) \exp\left\{a_i(k) - b_i(k)(y_i(k))^{\theta_{ii}} + \sum_{j=1,j\neq i}^{n} c_{ij}(k) \frac{(y_j(k))^{\theta_{ij}}}{d_{ij}(k) + (y_j(k))^{\theta_{ij}}}\right\}, i = 1, 2, \cdots, n.$$

We can easily see that $\{y_i(k)\}$ is a solution of system (1.1) and $m_i - \varepsilon \le y_i(k) \le M_i + \varepsilon$ for $k \in Z$. Since ε is an arbitrary small positive number, it follows that $m_i \le y_i(k) \le M_i$ and hence we complete the proof.

4 Stability of almost periodic solution

The main results of this paper concern the global stability of almost periodic solution of system (1.1) with condition (H1).

Theorem 4.1 Assume that (H1) and

(H2) $$\rho_i = \max\{|1-\theta_{ii}b_i^l m_i^{\theta_{ii}}|,\ |1-\theta_{ii}b_i^u M_i^{\theta_{ii}}|\} + \sum_{j=1,j\neq i}^{n} \frac{\theta_{ij}c_{ij}^u M_j^{\theta_{ij}}}{d_{ij}^l} < 1,\ i=1,2,\cdots,n,$$

hold. Then any positive solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) is globally attractive.
Proof. Assume that $(p_1(k), p_2(k), \cdots, p_n(k))$ is a solution of system (1.1) satisfying (H1). Let

$$x_i(k) = p_i(k)\exp\{u_i(k)\},\ \ i=1,2,\cdots,n.$$

Then system (1.1) is equivalent to

$$\begin{aligned} u_i(k+1) &= \ln x_i(k+1) - \ln p_i(k+1) \\ &= \ln x_i(k) + a_i(k) - b_i(k)(x_i(k))^{\theta_{ii}} + \sum_{j=1,j\neq i}^{n} c_{ij}(k)\frac{(x_j(k))^{\theta_{ij}}}{d_{ij}(k)+(x_j(k))^{\theta_{ij}}} \\ &\quad -\ln p_i(k) - a_i(k) + b_i(k)(p_i(k))^{\theta_{ii}} - \sum_{j=1,j\neq i}^{n} c_{ij}(k)\frac{(p_j(k))^{\theta_{ij}}}{d_{ij}(k)+(p_j(k))^{\theta_{ij}}} \\ &= u_i(k) - b_i(k)[(x_i(k))^{\theta_{ii}} - (p_i(k))^{\theta_{ii}}] \\ &\quad + \sum_{j=1,j\neq i}^{n} \frac{d_{ij}(k)c_{ij}(k)[(x_j(k))^{\theta_{ij}} - (p_j(k))^{\theta_{ij}}]}{[d_{ij}(k)+(x_j(k))^{\theta_{ij}}][d_{ij}(k)+(p_j(k))^{\theta_{ij}}]} \\ &= u_i(k) - b_i(k)(p_i(k))^{\theta_{ii}}\left[(\exp\{u_i(k)\})^{\theta_{ii}} - 1\right] \\ &\quad + \sum_{j=1,j\neq i}^{n} \frac{d_{ij}(k)c_{ij}(k)(p_j(k))^{\theta_{ij}}[(\exp\{u_j(k)\})^{\theta_{ij}} - 1]}{[d_{ij}(k)+(x_j(k))^{\theta_{ij}}][d_{ij}(k)+(p_j(k))^{\theta_{ij}}]},\ \ i=1,2,\cdots,n. \end{aligned}$$

Therefore,

$$\begin{aligned} u_i(k+1) &= u_i(k)\big(1-\theta_{ii}b_i(k)[p_i(k)\exp\{\lambda_i(k)u_i(k)\}]^{\theta_{ii}}\big) \\ &\quad + \sum_{j=1,j\neq i}^{n} \frac{d_{ij}(k)\theta_{ij}c_{ij}(k)u_j(k)[p_j(k)\exp\{\overline{\lambda_j}(k)u_j(k)\}]^{\theta_{ij}}}{[d_{ij}(k)+(x_j(k))^{\theta_{ij}}][d_{ij}(k)+(p_j(k))^{\theta_{ij}}]},\ \ i=1,2,\cdots,n, \end{aligned} \quad (4.1)$$

where $\lambda_i(k), \overline{\lambda_j}(k) \in [0,1]$. To complete the proof, it suffices to show that

$$\lim_{k\to+\infty} u_i(k) = 0,\ \ i=1,2,\cdots,n. \quad (4.2)$$

In view of (H2), we can choose $\varepsilon > 0$ such that

$$\rho_i^{\varepsilon} = \max\{|1-\theta_{ii}b_i^l(m_i-\varepsilon)^{\theta_{ii}}|,\ |1-\theta_{ii}b_i^u(M_i+\varepsilon)^{\theta_{ii}}|\} + \sum_{j=1,j\neq i}^{n} \frac{\theta_{ij}c_{ij}^u(M_j+\varepsilon)^{\theta_{ij}}}{d_{ij}^l} < 1,\ i=1,2,\cdots,n.$$

Let $\rho = \max\{\rho_i^{\varepsilon}\}$, then $\rho < 1$. According to Theorem 3.2, there exists a positive integer $k_0 \in Z^+$ such that

$$m_i - \varepsilon \le x_i(k) \le M_i + \varepsilon,\ \ m_i - \varepsilon \le p_i(k) \le M_i + \varepsilon,\ \ i=1,2,\cdots,n$$

for $k \ge k_0$.

Notice that $\lambda_i(k) \in [0,1]$ implies that $p_i(k)\exp\{\lambda_i(k)u_i(k)\}$ lies between $p_i(k)$ and $x_i(k)$, $\overline{\lambda_j}(k) \in [0,1]$ implies that $p_j(k)\exp\{\overline{\lambda_j}(k)u_j(k)\}$ lies between $p_j(k)$ and $x_j(k)$. From (4.1), we get

$$\begin{aligned} |u_i(k+1)| &\le \max\{|1-\theta_{ii}b_i^l(m_i-\varepsilon)^{\theta_{ii}}|,\ |1-\theta_{ii}b_i^u(M_i+\varepsilon)^{\theta_{ii}}|\}|u_i(k)| \\ &\quad + \sum_{j=1,j\neq i}^{n} \frac{\theta_{ij}c_{ij}^u(M_j+\varepsilon)^{\theta_{ij}}}{d_{ij}^l}|u_j(k)|,\ i=1,2,\cdots,n, \end{aligned} \quad (4.3)$$

for $k \geq k_0$.

In view of (4.3), we get

$$\max_{1\leq i\leq n} |u_i(k+1)| \leq \rho \max_{1\leq i\leq n} |u_i(k)|, \quad k \geq k_0.$$

This implies

$$\max_{1\leq i\leq n} |u_i(k)| \leq \rho^{k-k_0} \max_{1\leq i\leq n} |u_i(k_0)|, \quad k \geq k_0.$$

Then (4.2) holds and we can obtain

$$\lim_{k\to+\infty} |x_i(k) - p_i(k)| = 0, \quad i = 1, 2, \cdots, n. \tag{4.4}$$

Therefore, positive solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) is globally attractive. □

Theorem 4.2 Assume that (H1)-(H2) hold. Then system (1.1) admits a unique almost periodic solution which is globally attractive.

Proof. It follows from Proposition 3.3 that there exists a solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of system (1.1) satisfying $m_i \leq x_i(k) \leq M_i, k \in Z^+$.

Suppose that $(x_1(k), x_2(k), \cdots, x_n(k))$ is any solution of system (1.1), then there exists an integer valued sequence $\{k_p^{'}\}$, $k_p^{'} \to +\infty$ as $p \to +\infty$, such that $(x_1(k+k_p^{'}), x_2(k+k_p^{'}), \cdots, x_n(k+k_p^{'}))$ is a solution of the following system

$$x_i(k+1) = x_i(k)\exp\left\{a_i(k+k_p^{'}) - b_i(k+k_p^{'})(x_i(k))^{\theta_{ii}} + \sum_{j=1,j\neq i}^{n} c_{ij}(k+k_p^{'})\frac{(x_j(k))^{\theta_{ij}}}{d_{ij}(k+k_p^{'}) + (x_j(k))^{\theta_{ij}}}\right\},$$
$$i = 1, 2, \cdots, n.$$

From above discussion and Theorem 3.2, we have that not only $(x_1(k+k_p^{'}), x_2(k+k_p^{'}), \cdots, x_n(k+k_p^{'}))$ but also $(\Delta x_1(k+k_p^{'}), \Delta x_2(k+k_p^{'}), \cdots, \Delta x_n(k+k_p^{'}))$ are uniformly bounded, thus $(x_1(k+k_p^{'}), x_2(k+k_p^{'}), \cdots, x_n(k+k_p^{'}))$ are uniformly bounded and equi-continuous. By Ascoli's theorem[36], there exists a uniformly convergent subsequence $(x_1(k+k_p), x_2(k+k_p), \cdots, x_n(k+k_p)) \subseteq (x_1(k+k_p^{'}), x_2(k+k_p^{'}), \cdots, x_n(k+k_p^{'}))$ such that for any $\varepsilon > 0$, there exists a $k_0(\varepsilon) > 0$ with the property that if $m, n \geq k_0(\varepsilon)$ then

$|x_i(k+k_m) - x_i(k+k_n)| < \varepsilon,$

which shows from Lemma 2.3 that $(x_1(k), x_2(k), \cdots, x_n(k))$ is asymptotically almost periodic sequence. Thus, by Definition 2.2, we can express it as

$$x_i(k) = p_i(k) + q_i(k),$$

$i = 1, 2, \cdots, n$, where $\{p_i(k)\}$ is almost periodic in $k \in Z$ and $q_i(k) \to 0$ as $k \to +\infty$. In the following we show that $\{(p_1(k), p_2(k), \cdots, p_n(k))\}$ is an almost periodic solution of system (1.1).

From the properties of an almost periodic sequence, there exists an integer valued sequence $\{\delta_p\}$, $\delta_p \to +\infty$ as $p \to +\infty$, such that

$$a_i(k+\delta_p) \to a_i(k), \ \ b_i(k+\delta_p) \to b_i(k), \ \ c_{ij}(k+\delta_p) \to c_{ij}(k), \ \ d_{ij}(k+\delta_p) \to d_{ij}(k), \ \text{ as } \ p \to +\infty.$$

It is easy to know that $x_i(k+\delta_p) \to p_i(k)$, as $p \to \infty$, then we have

$$\begin{aligned} p_i(k+1) &= \lim_{p\to\infty} x_i(k+1+\delta_p) \\ &= \lim_{p\to\infty} x_i(k+\delta_p)\exp\Bigg\{a_i(k+\delta_p) - b_i(k+\delta_p)(x_i(k+\delta_p))^{\theta_{ii}} \\ &\quad + \sum_{j=1,j\neq i}^{n} c_{ij}(k+\delta_p)\frac{(x_j(k+\delta_p))^{\theta_{ij}}}{d_{ij}(k+\delta_p)+(x_j(k+\delta_p))^{\theta_{ij}}}\Bigg\} \\ &= \lim_{p\to\infty} p_i(k)\exp\Bigg\{a_i(k) - b_i(k)(p_i(k))^{\theta_{ii}} + \sum_{j=1,j\neq i}^{n} c_{ij}(k)\frac{(p_j(k))^{\theta_{ij}}}{d_{ij}(k)+(p_j(k))^{\theta_{ij}}}\Bigg\}. \end{aligned}$$

This prove that $p(k) = \{(p_1(k), p_2(k), \cdots, p_n(k))\}$ satisfied system (1.1), and $p(k)$ is a positive almost periodic solution of system (1.1).

Now, we show that there is only one positive almost periodic solution of system (1.1). For any two positive almost periodic solutions $(p_1(k), p_2(k), \cdots, p_n(k))$ and $(z_1(k), z_2(k), \cdots, z_n(k))$ of system (1.1), we claim that $p_i(k) = z_i(k), (i = 1, 2, \cdots, n)$ for all $k \in$ **Z**$^+$. Otherwise there must be at least one positive integer $K^* \in$ **Z**$^+$ such that $p_i(K^*) \neq z_i(K^*)$ for a certain positive integer i, i.e., $\Omega = |p_i(K^*) - z_i(K^*)| > 0$. So we can easily know that

$$\Omega = |\lim_{p\to+\infty} p_i(K^*+\delta_p) - \lim_{p\to+\infty} z_i(K^*+\delta_p)| = \lim_{p\to+\infty} |p_i(K^*+\delta_p) - z_i(K^*+\delta_p)| = \lim_{k\to+\infty} |p_i(k) - z_i(k)| > 0,$$

which is a contradiction to (4.4). Thus $p_i(k) = z_i(k)(i = 1, 2, \cdots, n)$ hold for $\forall k \in$ **Z**$^+$. Therefore, system (1.1) admits a unique almost periodic solution which is globally attractive. This completes the proof of Theorem 4.2. □

Remark 4.3 If $\theta_{ii} = \theta_{ij} = 1$, for system (1.1), the condition (H2) can be simplified. Therefore, we have the following result.

Corollary 4.4 Let $\theta_{ii} = \theta_{ij} = 1$. Assume that (H1) and

$$\rho_i = \max\{|1 - b_i^l m_i|,\ |1 - b_i^u M_i|\} + \sum_{j=1,j\neq i}^{n} \frac{c_{ij}^u M_j}{d_{ij}^l} < 1,\ i = 1, 2, \cdots, n,$$

hold. Then system (1.1) admits a unique almost periodic solution which is globally attractive.

In the following, the main results concern the existence of a unique uniformly asymptomatically stable almost periodic solution of system (1.1) by constructing a non-negative Lyapunov function.

Theorem 4.5 Assume that the condition (H1) hold, moreover, $0 < \beta < 1$, where

$$\begin{aligned} &\beta = \min_{1\le i\le n}\{\beta_i\}, \\ &\beta_i = 2\theta_{ii}b_i^l m_i - \theta_{ii}^2 b_i^{u2} M_i^2 - \sum_{j=1,j\neq i}^{n}\Big[\theta_{ji}^2 c_{ji}^{u2} + (1+2\theta_{ii}b_i^u M_i)\theta_{ij}c_{ij}^u + (1+2\theta_{jj}b_j^u M_j)\theta_{ji}c_{ji}^u \\ &\qquad + \sum_{l=1,l\neq i,j}^{n} \theta_{li}\theta_{lj}c_{li}^u c_{lj}^u\Big], \end{aligned}$$

$i = 1, 2, \cdots, n$. Then there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k),$
$\cdots, x_n(k))$ of system (1.1) which is bounded by Ω for all $k \in N^+$.

Proof. Let $p_i(k) = \ln x_i(k)$, $i = 1, 2, \cdots, n$. From system (1.1), we have

$$p_i(k+1) = p_i(k) + a_i(k) - b_i(k)e^{\theta_{ii}p_i(k)} + \sum_{j=1, j\neq i}^{n} c_{ij}(k)\frac{e^{\theta_{ij}p_j(k)}}{d_{ij}(k) + e^{\theta_{ij}p_j(k)}}. \tag{4.5}$$

From Proposition 3.3, we know that system (4.5) have bounded solution $(p_1(k), p_2(k), \cdots, p_n(k))$ satisfying

$$\ln m_i \leq p_i(k) \leq \ln M_i, \quad i = 1, 2, \cdots, n, \quad k \in Z^+.$$

Hence, $|p_i(k)| \leq A_i$, where $A_i = \max\{|\ln m_i|, |\ln M_i|\}$, $i = 1, 2, \cdots, n$.

For $X \in R^n$, we define the norm $\|X\| = \sum_{i=1}^{n} |x_i|$.

Consider the product system of system (4.5)

$$\begin{cases} p_i(k+1) = p_i(k) + a_i(k) - b_i(k)e^{\theta_{ii}p_i(k)} + \sum_{j=1, j\neq i}^{n} c_{ij}(k)\frac{e^{\theta_{ij}p_j(k)}}{d_{ij}(k) + e^{\theta_{ij}p_j(k)}}, \\ q_i(k+1) = q_i(k) + a_i(k) - b_i(k)e^{\theta_{ii}q_i(k)} + \sum_{j=1, j\neq i}^{n} c_{ij}(k)\frac{e^{\theta_{ij}q_j(k)}}{d_{ij}(k) + e^{\theta_{ij}q_j(k)}}, \quad i = 1, 2, \cdots, n. \end{cases} \tag{4.6}$$

We assume that $Q = (p_1(k), p_2(k), \cdots, p_n(k))$, $W = (q_1(k), q_2(k), \cdots, q_n(k))$ are any two solutions of system (4.5) defined on $Z^+ \times S^*$; then, $\|Q\| \leq B$, $\|W\| \leq B$, where $B = \sum_{i=1}^{n} A_i$, $S^* = \{(p_1(k), p_2(k), \cdots, p_n(k)) | \ln m_i \leq p_i(n) \leq \ln M_i, i = 1, 2, \cdots, n, k \in Z^+\}$.

Let us construct a Lyapunov function defined on $Z^+ \times S^* \times S^*$ as follows:

$$V(k, Q, W) = \sum_{i=1}^{n} (p_i(k) - q_i(k))^2.$$

It is obvious that the norm $\|Q - W\| = \sum_{i=1}^{n} |p_i(k) - q_i(k)|$ is equivalent to $\|Q - W\|_* = [\sum_{i=1}^{n} \{(p_i(k) - q_i(k))^2]^{1/2}$; that is, there are two constants $c_1 > 0, c_2 > 0$, such that

$$c_1\|Q - W\| \leq \|Q - W\|_* \leq c_2\|Q - W\|,$$

then,

$$(c_1\|Q - W\|)^2 \leq V(k, Q, W) \leq (c_2\|Q - W\|)^2.$$

Let $\psi, \varphi \in C(R^+, R^+), \psi(x) = c_1^2 x^2, \varphi(x) = c_2^2 x^2$; then, condition (i) of Theorem 2.7 is satisfied.

Moreover, for any $(k, Q, W), (k, \overline{Q}, \overline{W}) \in Z^+ \times S^* \times S^*$, we have

$$\begin{aligned}
&|V(k,Q,W) - V(k,\overline{Q},\overline{W})| \\
&= \Big| \sum_{i=1}^{n} (p_i(k) - q_i(k))^2 - \sum_{i=1}^{n} (\overline{p_i}(k) - \overline{q_i}(k))^2 \Big| \\
&\leq \sum_{i=1}^{n} |(p_i(k) - q_i(k))^2 - (\overline{p_i}(k) - \overline{q_i}(k))^2| \\
&= \sum_{i=1}^{n} |(p_i(k) - q_i(k)) + (\overline{p_i}(k) - \overline{q_i}(k))||(p_i(k) - q_i(k)) - (\overline{p_i}(k) - \overline{q_i}(k))| \\
&\leq \sum_{i=1}^{n} (|p_i(k)| + |q_i(k)| + |\overline{p_i}(k)| + |\overline{q_i}(k)|)(|p_i(k) - \overline{p_i}(k)| + |q_i(k) - \overline{q_i}(k)|) \\
&\leq L\big[\sum_{i=1}^{n} |p_i(k) - \overline{p_i}(k)| + \sum_{i=1}^{n} |q_i(k) - \overline{q_i}(k)|\big] \\
&= L(\|Q - \overline{Q}\| + \|W - \overline{W}\|),
\end{aligned}$$

where $\overline{Q} = (\overline{p_1}(k), \overline{p_2}(k), \cdots, \overline{p_n}(k))$, $\overline{W} = (\overline{q_1}(k), \overline{q_2}(k), \cdots, \overline{q_n}(k))$, and $L = 4 \max\limits_{1 \leq i \leq n} \{A_i\}$. Thus, condition (ii) of Theorem 2.7 is satisfied.

Finally, calculating the $\Delta V(k)$ of $V(k)$ along the solutions of system (4.6), we have

$$\begin{aligned}
&\Delta V_{(4.6)}(k) = V(k+1) - V(k) \\
&= \sum_{i=1}^{n} (p_i(k+1) - q_i(k+1))^2 - \sum_{i=1}^{n} (p_i(k) - q_i(k))^2 \\
&= \sum_{i=1}^{n} [(p_i(k+1) - q_i(k+1))^2 - (p_i(k) - q_i(k))^2] \\
&= \sum_{i=1}^{n} \Bigg\{ \Bigg[(p_i(k) - q_i(k)) - b_i(k)(e^{\theta_{ii} p_i(k)} - e^{\theta_{ii} q_i(k)}) + \sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k) d_{ij}(k)(e^{\theta_{ij} p_j(k)} - e^{\theta_{ij} q_j(k)})}{(d_{ij}(k) + e^{\theta_{ij} p_j(k)})(d_{ij}(k) + e^{\theta_{ij} q_j(k)})} \Bigg]^2 \\
&\quad - (p_i(k) - q_i(k))^2 \Bigg\} \\
&= \sum_{i=1}^{n} \Bigg\{ -2b_i(k)(p_i(k) - q_i(k))(e^{\theta_{ii} p_i(k)} - e^{\theta_{ii} q_i(k)}) + b_i^2(k)(e^{\theta_{ii} p_i(k)} - e^{\theta_{ii} q_i(k)})^2 \\
&\quad - 2b_i(k)(e^{\theta_{ii} p_i(k)} - e^{\theta_{ii} q_i(k)}) \sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k) d_{ij}(k)(e^{\theta_{ij} p_j(k)} - e^{\theta_{ij} q_j(k)})}{(d_{ij}(k) + e^{\theta_{ij} p_j(k)})(d_{ij}(k) + e^{\theta_{ij} q_j(k)})} \\
&\qquad + \Bigg(\sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k) d_{ij}(k)(e^{\theta_{ij} p_j(k)} - e^{\theta_{ij} q_j(k)})}{(d_{ij}(k) + e^{\theta_{ij} p_j(k)})(d_{ij}(k) + e^{\theta_{ij} q_j(k)})} \Bigg)^2 \\
&\qquad + 2(p_i(k) - q_i(k)) \sum_{j=1, j\neq i}^{n} \frac{c_{ij}(k) d_{ij}(k)(e^{\theta_{ij} p_j(k)} - e^{\theta_{ij} q_j(k)})}{(d_{ij}(k) + e^{\theta_{ij} p_j(k)})(d_{ij}(k) + e^{\theta_{ij} q_j(k)})} \Bigg\}.
\end{aligned}$$

By the mean value theorem, it derives that

$$\mathrm{e}^{\theta_{ij} p_j(k)} - \mathrm{e}^{\theta_{ij} q_j(k)} = \theta_{ij} \xi_{ij}(k)(p_j(k) - q_j(k)),\ i, j = 1, 2, \cdots, n,$$

where $\xi_{ij}(k)$ lies between $e^{\theta_{ij}p_j(k)}$ and $e^{\theta_{ij}q_j(k)}$. Then, we have

$$\begin{aligned}
\Delta V_{(4.6)}(k) &= \sum_{i=1}^{n}\Big\{ -2\theta_{ii}b_i(k)\xi_{ii}(k)(p_i(k)-q_i(k))^2 + \theta_{ii}^2 b_i^2(k)\xi_{ii}^2(k)(p_i(k)-q_i(k))^2 \\
&\quad - 2\theta_{ii}b_i(k)\xi_{ii}(k)(p_i(k)-q_i(k)) \sum_{j=1,j\neq i}^{n} \frac{\theta_{ij}c_{ij}(k)d_{ij}(k)\xi_{ij}(k)(p_j(k)-q_j(k))}{(d_{ij}(k)+e^{\theta_{ij}p_j(k)})(d_{ij}(k)+e^{\theta_{ij}q_j(k)})} \\
&\quad + \Big(\sum_{j=1,j\neq i}^{n} \frac{\theta_{ij}c_{ij}(k)d_{ij}(k)\xi_{ij}(k)(p_j(k)-q_j(k))}{(d_{ij}(k)+e^{\theta_{ij}p_j(k)})(d_{ij}(k)+e^{\theta_{ij}q_j(k)})}\Big)^2 \\
&\quad + 2(p_i(k)-q_i(k)) \sum_{j=1,j\neq i}^{n} \frac{\theta_{ij}c_{ij}(k)d_{ij}(k)\xi_{ij}(k)(p_j(k)-q_j(k))}{(d_{ij}(k)+e^{\theta_{ij}p_j(k)})(d_{ij}(k)+e^{\theta_{ij}q_j(k)})}\Big\} \\
&= \sum_{i=1}^{n}\Big\{\Big(-2\theta_{ii}b_i(k)\xi_{ii}(k) + \theta_{ii}^2 b_i^2(k)\xi_{ii}^2(k) \\
&\qquad + \sum_{j=1,j\neq i}^{n} \frac{\theta_{ji}^2 c_{ji}^2(k)d_{ji}^2(k)\xi_{ji}^2(k)}{(d_{ji}(k)+e^{\theta_{ji}p_i(k)})^2(d_{ji}(k)+e^{\theta_{ji}q_i(k)})^2}\Big)(p_i(k)-q_i(k))^2 \\
&\quad +2\sum_{j=1,j\neq i}^{n}\Big(\frac{[1-2\theta_{ii}b_i(k)\xi_{ii}(k)]\theta_{ij}c_{ij}(k)d_{ij}(k)\xi_{ij}(k)}{(d_{ij}(k)+e^{\theta_{ij}p_j(k)})(d_{ij}(k)+e^{\theta_{ij}q_j(k)})} \\
&\qquad +\frac{1}{2}\sum_{l=1,l\neq i,j}^{n} \frac{\theta_{li}c_{li}(k)d_{li}(k)\xi_{li}(k)}{(d_{li}(k)+e^{\theta_{li}p_i(k)})(d_{li}(k)+e^{\theta_{li}q_i(k)})} \frac{\theta_{lj}c_{lj}(k)d_{lj}(k)\xi_{lj}(k)}{(d_{lj}(k)+e^{\theta_{lj}p_j(k)})(d_{lj}(k)+e^{\theta_{lj}q_j(k)})}\Big)\times \\
&\qquad (p_i(k)-q_i(k))(p_j(k)-q_j(k))\Big\} \\
&\le \sum_{i=1}^{n}\Big\{\Big(-2\theta_{ii}b_i(k)\xi_{ii}(k) + \theta_{ii}^2 b_i^2(k)\xi_{ii}^2(k) \\
&\qquad + \sum_{j=1,j\neq i}^{n} \frac{\theta_{ji}^2 c_{ji}^2(k)d_{ji}^2(k)\xi_{ji}^2(k)}{(d_{ji}(k)+e^{\theta_{ji}p_i(k)})^2(d_{ji}(k)+e^{\theta_{ji}q_i(k)})^2}\Big)(p_i(k)-q_i(k))^2 \\
&\quad +2\Big|\sum_{j=1,j\neq i}^{n}\Big(\frac{[1-2\theta_{ii}b_i(k)\xi_{ii}(k)]\theta_{ij}c_{ij}(k)d_{ij}(k)\xi_{ij}(k)}{(d_{ij}(k)+e^{\theta_{ij}p_j(k)})(d_{ij}(k)+e^{\theta_{ij}q_j(k)})} \\
&\qquad +\frac{1}{2}\sum_{l=1,l\neq i,j}^{n} \frac{\theta_{li}c_{li}(k)d_{li}(k)\xi_{li}(k)}{(d_{li}(k)+e^{\theta_{li}p_i(k)})(d_{li}(k)+e^{\theta_{li}q_i(k)})} \frac{\theta_{lj}c_{lj}(k)d_{lj}(k)\xi_{lj}(k)}{(d_{lj}(k)+e^{\theta_{lj}p_j(k)})(d_{lj}(k)+e^{\theta_{lj}q_j(k)})}\Big)\times \\
&\qquad (p_i(k)-q_i(k))(p_j(k)-q_j(k))\Big|\Big\}.
\end{aligned}$$

Then, we have

$$\Delta V_{(4.2)}(k) \le \sum_{i=1}^{n}[V_{i1}(k)+V_{i2}(k)],$$

where

$$\begin{aligned}
V_{i1}(k) &= \Big(-2\theta_{ii}b_i(k)\xi_{ii}(k) + \theta_{ii}^2 b_i^2(k)\xi_{ii}^2(k) + \sum_{j=1,j\neq i}^{n} \frac{\theta_{ji}^2 c_{ji}^2(k)d_{ji}^2(k)\xi_{ji}^2(k)}{(d_{ji}(k)+e^{\theta_{ji}p_i(k)})^2(d_{ji}(k)+e^{\theta_{ji}q_i(k)})^2}\Big)(p_i(k)-q_i(k))^2 \\
&\le \Big(-2\theta_{ii}b_i^l m_i + \theta_{ii}^2 b_i^{u2} M_i^2 + \sum_{j=1,j\neq i}^{n} \theta_{ji}^2 c_{ji}^{u2}\Big)(p_i(k)-q_i(k))^2,
\end{aligned}$$

$$V_{i2}(k) = 2\Bigg|\sum_{j=1,j\neq i}^{n}\Bigg(\frac{(1-2\theta_{ii}b_i(k)\xi_{ii}(k))\theta_{ij}c_{ij}(k)d_{ij}(k)\xi_{ij}(k)}{(d_{ij}(k)+e^{\theta_{ij}p_j(k)})(d_{ij}(k)+e^{\theta_{ij}q_j(k)})}$$
$$+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}\frac{\theta_{li}c_{li}(k)d_{li}(k)\xi_{li}(k)}{(d_{li}(k)+e^{p_i(k)})(d_{li}(k)+e^{q_i(k)})}\frac{\theta_{lj}c_{lj}(k)d_{lj}(k)\xi_{lj}(k)}{(d_{lj}(k)+e^{\theta_{lj}p_j(k)})(d_{lj}(k)+e^{\theta_{lj}q_j(k)})}\Bigg)\times$$
$$(p_i(k)-q_i(k))(p_j(k)-q_j(k))\Bigg|$$
$$\leq \sum_{j=1,j\neq i}^{n}\Bigg((1+2\theta_{ii}b_i^uM_i)\theta_{ij}c_{ij}^u+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}\theta_{li}\theta_{lj}c_{li}^uc_{lj}^u\Bigg)[(p_i(k)-q_i(k))^2+(p_j(k)-q_j(k))^2].$$

Hence, we have

$$\Delta V_{(4.6)}(k) \leq \sum_{i=1}^{n}\Bigg\{\Bigg(-2\theta_{ii}b_i^lm_i+\theta_{ii}^2b_i^{u2}M_i^2$$
$$+\sum_{j=1,j\neq i}^{n}\Bigg[\theta_{ji}^2c_{ji}^{u2}+(1+2\theta_{ii}b_i^uM_i)\theta_{ij}c_{ij}^u+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}\theta_{li}\theta_{lj}c_{li}^uc_{lj}^u\Bigg]\Bigg)(p_i(k)-q_i(k))^2$$
$$+\sum_{j=1,j\neq i}^{n}\Bigg((1+2\theta_{ii}b_i^uM_i)\theta_{ij}c_{ij}^u+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}\theta_{li}\theta_{lj}c_{li}^uc_{lj}^u\Bigg)(p_j(k)-q_j(k))^2\Bigg\}$$
$$=\sum_{i=1}^{n}\Bigg\{\Bigg(-2\theta_{ii}b_i^lm_i+\theta_{ii}^2b_i^{u2}M_i^2+\sum_{j=1,j\neq i}^{n}\Bigg[\theta_{ji}^2c_{ji}^{u2}+(1+2\theta_{ii}b_i^uM_i)\theta_{ij}c_{ij}^u$$
$$+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}\theta_{li}\theta_{lj}c_{li}^uc_{lj}^u\Bigg]\Bigg)(p_i(k)-q_i(k))^2$$
$$+\sum_{j=1,j\neq i}^{n}\Bigg((1+2\theta_{jj}b_j^uM_j)\theta_{ji}c_{ji}^u+\frac{1}{2}\sum_{l=1,l\neq i,j}^{n}\theta_{li}\theta_{lj}c_{li}^uc_{lj}^u\Bigg)(p_i(k)-q_i(k))^2\Bigg\}$$
$$=\sum_{i=1}^{n}\Bigg\{\Bigg(-2\theta_{ii}b_i^lm_i+\theta_{ii}^2b_i^{u2}M_i^2+\sum_{j=1,j\neq i}^{n}\Bigg[\theta_{ji}^2c_{ji}^{u2}+(1+2\theta_{ii}b_i^uM_i)\theta_{ij}c_{ij}^u+(1+2\theta_{jj}b_j^uM_j)\theta_{ji}c_{ji}^u$$
$$+\sum_{l=1,l\neq i,j}^{n}\theta_{li}\theta_{lj}c_{li}^uc_{lj}^u\Bigg]\Bigg)(p_i(k)-q_i(k))^2\Bigg\}$$
$$=-\sum_{i=1}^{n}\Bigg\{\Bigg(2\theta_{ii}b_i^lm_i-\theta_{ii}^2b_i^{u2}M_i^2-\sum_{j=1,j\neq i}^{n}\Bigg[\theta_{ji}^2c_{ji}^{u2}+(1+2\theta_{ii}b_i^uM_i)\theta_{ij}c_{ij}^u+(1+2\theta_{jj}b_j^uM_j)\theta_{ji}c_{ji}^u$$
$$+\sum_{l=1,l\neq i,j}^{n}\theta_{li}\theta_{lj}c_{li}^uc_{lj}^u\Bigg]\Bigg)(p_i(k)-q_i(k))^2\Bigg\}$$
$$\leq -\sum_{i=1}^{n}\beta_i(p_i(k)-q_i(k))^2$$
$$\leq -\beta\sum_{i=1}^{n}(p_i(k)-q_i(k))^2$$
$$=-\beta V(k,Q,W),$$

where $\beta=\min_{1\leq i\leq n}\{\beta_i\}$. That is, there exists a positive constant $0<\beta<1$ such that

$$\Delta V_{(4.6)}(k,Q,W)\leq -\beta V(k,Q,W).$$

From $0 < \beta < 1$, the condition (iii) of Theorem 2.7 is satisfied. So, according to Theorem 2.7, there exists a unique uniformly asymptotically stable almost periodic solution $(p_1(k), p_2(k), \cdots, p_n(k))$ of (4.5) which is bounded by S^* for all $k \in Z^+$. It means that there exists a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k), \cdots, x_n(k))$ of (1.1) which is bounded by Ω for all $k \in Z^+$. This completed the proof.

Remark 4.6 If $n = 2$, the conditions of Theorem 4.5 can be simplified. Therefore, we have the following result.

Corollary 4.7 Let $n = 2$, and assume further that $0 < \beta < 1$, where

$$\begin{aligned}
&\beta = \min\{\beta_{12}, \beta_{21}\},\\
&\beta_{ij} = 2\theta_{ii}b_i^l m_i - \theta_{ii}^2 b_i^{u2} M_i^2 - (1 + 2\theta_{ii}b_i^u M_i)\theta_{ij}c_{ij}^u - (1 + \theta_{ji}c_{ji}^u + 2\theta_{jj}b_j^u M_j)\theta_{ji}c_{ji}^u,
\end{aligned}$$

$i, j = 1, 2, j \neq i$. Then system (1.1) admits a unique uniformly asymptotically stable almost periodic solution $(x_1(k), x_2(k))$ which is bounded by Ω for all $k \in Z^+$.

5 Numerical Simulations

In this section, we give the following examples to check the feasibility of our results.
Example 5.1 Consider the discrete multispecies Gilpin-Ayala mutualism system:

$$\begin{cases}
x_1(k+1) = x_1(k)\exp\Big\{1.25 - 0.021\sin(\sqrt{2}k) - (1.25 + 0.014\sin(\sqrt{3}k))x_1^{\frac{1}{2}}(k)\\
\qquad\qquad +(0.02 + 0.002\cos(\sqrt{5}k))\dfrac{x_2^{\frac{1}{3}}(k)}{2 + x_2^{\frac{1}{3}}(k)} + (0.02 + 0.001\cos(\sqrt{2}k))\dfrac{x_3^{\frac{2}{3}}(k)}{1 + x_3^{\frac{2}{3}}(k)}\Big\},\\
x_2(k+1) = x_2(k)\exp\Big\{1.17 - 0.025\sin(\sqrt{3}k) + (0.02 + 0.003\sin(\sqrt{3}k))\dfrac{x_1(k)}{1 + x_1(k)}\\
\qquad\qquad -(1.11 + 0.015\sin(\sqrt{5}k))x_2^{\frac{1}{3}}(k) + (0.025 + 0.002\cos(\sqrt{5}k))\dfrac{x_3^{\frac{3}{2}}(k)}{2 + x_3^{\frac{3}{2}}(k)}\Big\},\\
x_3(k+1) = x_3(k)\exp\Big\{1.12 - 0.03\sin(\sqrt{3}k) + (0.03 + 0.0025\cos(\sqrt{2}k))\dfrac{x_1^{\frac{1}{2}}(k)}{1 + x_1^{\frac{1}{2}}(k)}\\
\qquad\qquad +(0.028 + 0.0015\cos(\sqrt{2}k))\dfrac{x_2^2(k)}{2 + x_2^2(k)} - (1.16 + 0.02\sin(\sqrt{3}n))x_3^{\frac{1}{4}}(k)\Big\}.
\end{cases} \tag{5.1}$$

A computation shows that

$$\rho_1 \approx 0.1051,\ \rho_2 \approx 0.0144,\ \rho_3 \approx 0.0613,$$

that $\max\{\rho_1, \rho_2, \rho_3\} < 1$. Hence, there exists a unique globally attractive almost periodic solution of system (5.1). Our numerical simulations support our results(see Figs.1,2 and 3).

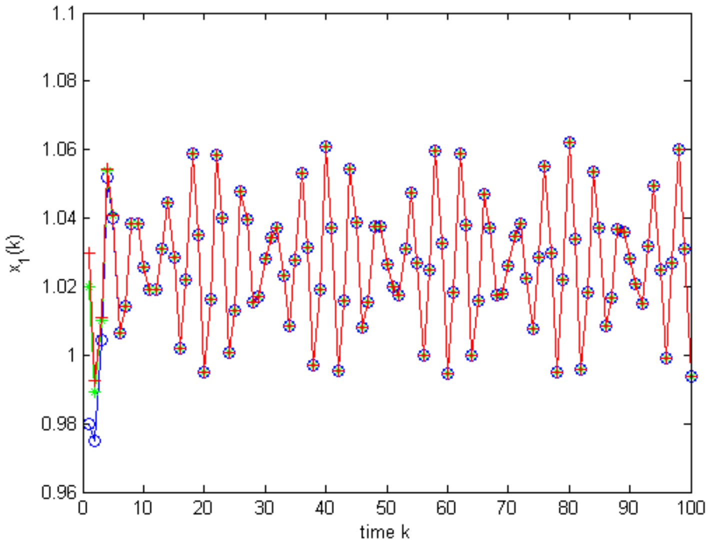

Figure1: Dynamic behavior of the first component $x_1(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (5.1) with the initial conditions (0.98,1.27,0.99), (1.02,1.2,1.05) and (1.03,1.17,0.93) for $k \in [1, 100]$, respectively.

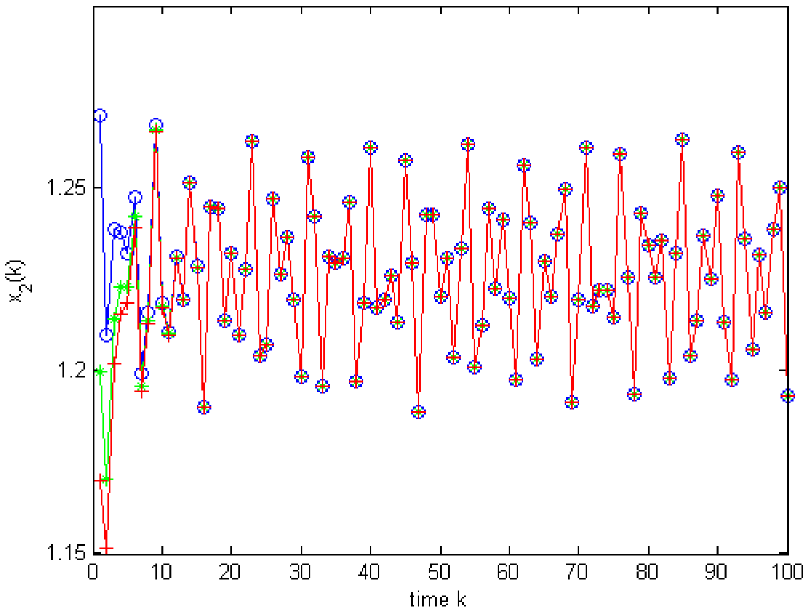

Figure 2: Dynamic behavior of the second component $x_2(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (5.1) with the initial conditions (0.98,1.27,0.99), (1.02,1.2,1.05) and (1.03,1.17,0.93) for $k \in [1, 100]$, respectively.

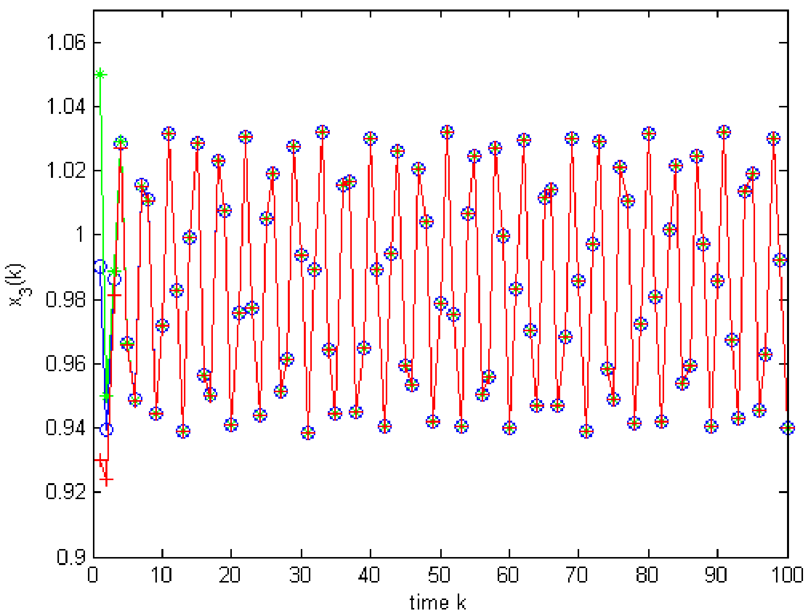

Figure 3: Dynamic behavior of the third component $x_3(k)$ of the solution $(x_1(k), x_2(k), x_3(k))$ to system (5.1) with the initial conditions (0.98,1.27,0.99), (1.02,1.2,1.05) and (1.03,1.17,0.93) for $k \in [1, 100]$, respectively.

Example 5.2 Consider the discrete Gilpin-Ayala mutualism system:

$$\begin{cases} x_1(k+1) = x_1(k)\exp\Big\{1.15 - 0.01\sin(\sqrt{2}k) - (1.16 - 0.02\cos(\sqrt{3}k))x_1^{\frac{1}{2}}(k) \\ \qquad\qquad +(0.05 - 0.002\cos(\sqrt{3}k))\dfrac{x_2^{\frac{1}{4}}(k)}{1 + x_2^{\frac{1}{4}}(k)}\Big\}, \\ x_2(k+1) = x_2(k)\exp\Big\{1.25 - 0.025\sin(\sqrt{3}k) + (0.02 - 0.0025\cos(\sqrt{5}k))\dfrac{x_1^{\frac{1}{3}}(k)}{2 + x_1^{\frac{1}{3}}(k)} \\ \qquad\qquad -(1.1 - 0.02\sin(\sqrt{2}k))x_2^{\frac{1}{3}}(k)\Big\}. \end{cases} \tag{5.2}$$

A computation shows that

$$\beta_{12} \approx 0.0143, \ \beta_{21} \approx 0.0632,$$

that $\min\{\beta_{12}, \beta_{21}\} < 1$. Hence, there exists a unique uniformly asymptotically stable almost periodic solution of system (5.2). Our numerical simulations support our results(see Figs.4 and 5).

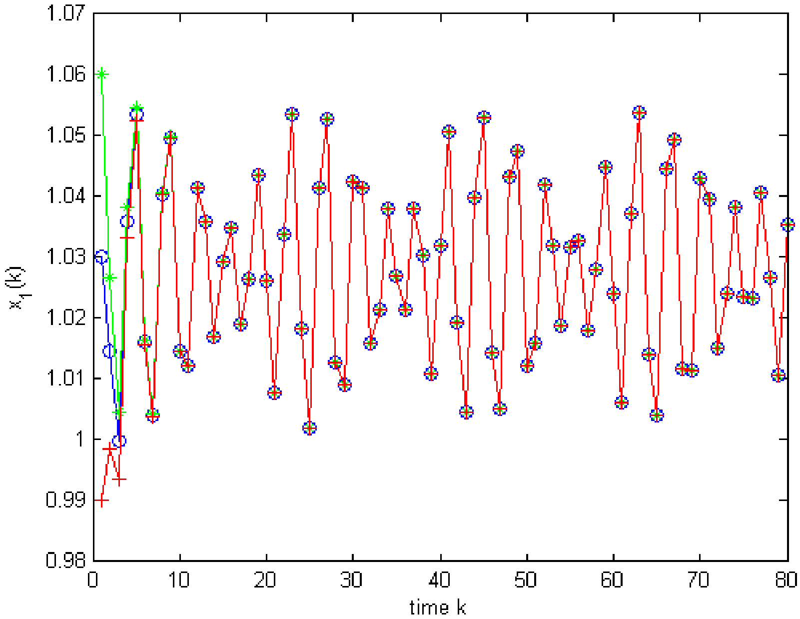

Figure 4: Dynamic behavior of the first component $x_1(k)$ of the solution $(x_1(k), x_2(k))$ to system (5.2) with the initial conditions (1.03,1.42), (1.06,1.51) and (0.99,1.58) for $k \in [1, 80]$, respectively.

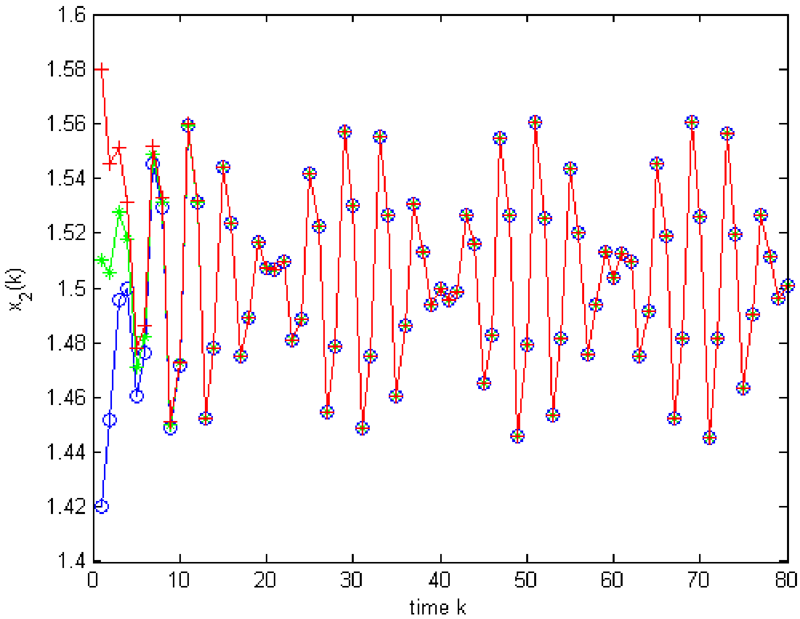

Figure 5: Dynamic behavior of the second component $x_2(k)$ of the solution $(x_1(k), x_2(k))$ to system (5.2) with the initial conditions (1.03,1.42), (1.06,1.51) and (0.99,1.58) for $k \in [1, 80]$, respectively.

6 Conclusion

In this paper, assuming that the coefficients in system (1.1) are bounded non-negative almost periodic sequences, we obtain the sufficient conditions for the existence of a unique almost periodic solution which is globally attractive. By comparative analysis, we find that when the coefficients in system (1.1) are almost periodic, the existence of a unique almost periodic solution of system (1.1) is determined by the global attractivity of system (1.1), which implies that there is no additional condition to add.

Furthermore, for the almost periodic discrete multispecies Gilpin-Ayala mutualism system (1.1) with time delays or feedback controls, we would like to mention here the question of how to study the almost periodicity of the system and whether the existence of a unique almost periodic solution is determined by the global attractivity of the system or not. It is, in fact, a very challenging problem, and we leave it for our future work.

Acknowledgment

The authors are grateful to the anonymous referees for their excellent suggestions, which greatly improved the presentation of the paper. Also, the authors declare that there is no conflict of interests regarding the publication of this paper, and there are no financial interest conflicts between the authors and the commercial identity. This work was partially supported by Shaanxi Provincial Education Department of China(no.2013JK1098). There are no financial interest conflicts between the authors and the commercial identity.

COMPETING INTERESTS

The author declares that no competing interests exist.

References

[1] Wolin CL, Lawlor LR. Models of facultative mutualism: Density effects. The American Naturalist. 1984;124:843-862.

[2] Yonghui Xia, Jinde Cao, Sui Sun Cheng. Periodic solutions for a Lotka-Volterra mutualism system with several delays. Applied Mathematical Modelling. 2007;31:1960-1969.

[3] Yongkun Li, Hongtao Zhang. Existence of periodic solutions for a periodic mutualism model on time scales. Journal of Mathematical Analysis and Applications. 2008;343:818-825.

[4] Changyou Wang, Shu Wang, Fuping Yang, Linrui Li. Global asymptotic stability of positive equilibrium of three-species Lotka-Volterra mutualism models with diffusion and delay effects. Applied Mathematical Modelling. 2007;34:4278-4288.

[5] Hui Zhang, Yingqi Li, Bin Jing, Weizhou Zhao. Global stability of almost periodic solution of multispecies mutualism system with time delays and impulsive effects. Applied Mathematics and Computation. 2014;232:1138-1150.

[6] Yongkun Li, Hongtao Zhang. Existence of periodic solutions for a periodic mutualism model on time scales. Journal of Mathematical Analysis and Application. 2008;343:818-825.

[7] Yongzhi Liao, Tianwei Zhang. Almost Periodic Solutions of a discrete mutualism model with variable delays. Discrete Dynamics in Nature and Society; 2012. Article ID 742102, 27 pages.

[8] Hui Zhang, Yingqi Li, Bin Jing. Global attractivity and almost periodic solution of a discrete mutualism model with delays. Mathematical Methods in the Applied Science. 2014;37:3013-3025.

[9] Hui Zhang, Bin Jing, Yingqi Li, Xiaofeng Fang. Global analysis of almost periodic solution of a discrete multispecies mutualism system. Journal of Applied Mathematics. Volume 2014, Article ID 107968, 12 pages.

[10] Hui Zhang, Feng Feng, Bin Jing, Yingqi Li. Almost periodic solution of a multispecies discrete mutualism system with feedback controls. Discrete Dynamics in Nature and Society. Volume 2015, Article ID 268378, 14 pages.

[11] Gopalsamy K. Global asymptotical stability in a periodic Lotka-Volterra system. J. Austral. Math. Soc. Ser. B. 1985;29:66-72.

[12] Tang X, Zou X. On positive periodic solutions of Lotka-Volterra competition systems with deviating arguments. Proc. Amer. Math. Soc. 2006;134:2967-2974.

[13] Tang X, Cao D, Zou X. Global attractivity of positive periodic solution to periodic Lotka-Volterra competition systems. Journal of Differential Equations. 2006;228:580-610.

[14] Mengxin He, Fengde Chen. Dynamic behaviors of the impulsive periodic multi-species predator-prey system. Computers and Mathematics with Applications. 2009;57:248-265.

[15] Wensheng Yang, Xuepeng Li. Permanence of a discrete nonlinear N-species cooperation system with time delays and feedback controls. Applied Mathematics and Computation. 2011;218:3581-3586.

[16] Zuowei Cai, Lihong Huang, Haibo Chen. Positive periodic solution for a multispecies competition-predator system with Holling III functional response and time delays. Applied Mathematics and Computation. 2011;217:4866-4878.

[17] Gilpin ME, Ayala FJ. Global model of growth and competition. Proc. Natl. Acad. Sci. USA. 1973;70:3590-3593.

[18] Ayala FJ, Gilpin ME, Eherenfeld JG. Competition between species: Theoretical models and experimental tests. Theor. Popul. Biol. 1973;4:331-356.

[19] Meng Fan, Ke Wang. Global periodic solutions of a generalized n-species GilpinCAyala competition model. Computers and Mathematics with Applications. 2000;40:1141-1151.

[20] Fengde Chen, Liping Wu, Zhong Li. Permanence and global attractivity of the discrete Gilpin-Ayala type population model. Computers and Mathematics with Applications. 2007;53:1214-1227.

[21] Zhong Li, Fengde Chen. Extinction and almost periodic solutions of a discrete Gilpin-Ayala type population model. Journal of Difference Equations and Applications. 2013;19:719-737.

[22] Qinglong Wang, Zhijun Liu. Uniformly Asymptotic Stability of Positive Almost Periodic Solutions for a Discrete Competitive System. Journal of Applied Mathematics. Volume 2013, Article ID 182158, 9 pages.

[23] Tianwei Zhang, Xiaorong Gan. Almost periodic solutions for a discrete fishing model with feedback control and time delays. Commun Nonlinear Sci Numer Simulat. 2014;19:150-163.

[24] Zengji Du, Yansen Lv. Permanence and almost periodic solution of a Lotka-Volterra model with mutual interference and time delays. Applied Mathematical Modelling. 2013;37:1054-1068.

[25] Wang Li, Mei Yu, Pengcheng Niu. Periodic solution and almost periodic solution of impulsive Lasota-Wazewska model with multiple time-varying delays. Computers and Mathematics with Applications. 2012;64:2383-2394.

[26] Bixiang Yang, Jianli Li. An almost periodic solution for an impulsive two-species logarithmic population model with time-varying delay. Mathematical and Computer Modelling. 2012;55:1963-1968.

[27] Alzabut JO, Stamovb GT, Sermutlu E. Positive almost periodic solutions for a delay logarithmic population model. Mathematical and Computer Modelling. 2011;53:161-167.

[28] Zhong Li, Maoan Han, Fengde Chen. Almost periodic solutions of a discrete almost periodic logistic equation with delay. Applied Mathematics and Computation. 2014;232:743-751.

[29]Yongkun Li, Tianwei Zhang. Permanence and almost periodic sequence solution for a discrete delay logistic equation with feedback control. Nonlinear Analysis: Real World Application. 2011;12:1850-1864.

[30] Fink AM, Seifert G. Liapunov functions and almost periodic solutions for almost periodic systems. Journal of Differential Equations. 1969;5L307-313.

[31] Rong Yuan. The existence of almost periodic solutions of retarded differential equations with piecewise constant argument. Nonlinear Anal. 2002;48:1013-1032.

[32] Liping Wu, Fengde Chen, Zhong Li. Permanence and global attrctivity of a discrete Schoener's competition model with delays. Mathematical and Computer Modelling. 2009;49:1607-1617.

[33] Samoilenko AM, Perestyuk NA. Impulsive differential equations in: world scientific series on nonlinear science. World Scientific, Singapore; 1995.

[34] Shunian Zhang, Zheng G. Almost periodic solutions of delay difference systems. Applied Mathematics and Computation. 2002;131:497-516.

[35] Shunian Zhang. Existence of almosti periodic solution for difference systems. Annals of Differential Equations. 2000;16:184-206.

[36] Agarwal RP, Bohner M, Rehak P. Half-linear dynamic equations. nonlinear analysis and applications to V. Laksh-mikantham on his 80th Birthday. Kluwer Academic Publishers, Dordrecht, 2003;1:1-57.

Peer-review history:
The peer review history for this paper can be accessed here:
http://sciencedomain.org/review-history/10158