British Journal of Mathematics & Computer Science
12(2): 1-4, 2016, Article no.BJMCS.21606
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

The Degree of a p-group G with $|G'| = p$

Mohammad Hassan Abbaspour[1*]

[1] *Faculty of Sciences, Khoy Branch, Islamic Azad University, Khoy, Iran.*

Article Information

DOI: 10.9734/BJMCS/2016/21606
Editor(s):
(1) Zuomao Yan, Department of Mathematics, Hexi University, China.
Reviewers:
(1) Oswaldo Gonzlez-Gaxiola, Universidad Autnoma Metropolitana, Mexico.
(2) S. T. Liu, Sichuan University of Science and Engineering, China.
(3) Francisco Bulnes, Tecnolgico de Estudios Superiores de Chalco, Mexico.
(4) D. Madhusudhana Rao, VSR and NVR College, Andhra Pradesh, India.
Complete Peer review History: http://sciencedomain.org/review-history/11711

Original Research Article

Received: 25 August 2015
Accepted: 20 September 2015
Published: 07 October 2015

Abstract

Let G be a finite group and $\delta(G)$ denote the least $n \in \mathbb{N}$ such that G is isomorphic to a subgroup of the symmetric group S_n. The number $\delta(G)$ is called the degree of G. This paper is a note on $\delta(G)$ for a finite p-group G with derived subgroup of order p. Our method is based on quasi-permutation representations of a finite group.

Keywords: Degree of a finite group; quasi-permutation representation; character theory.

2010 Mathematics Subject Classification: 20C15; 20B05.

1 Introduction

By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G, then $\delta(G)$ is the minimal degree of a faithful permutation representation of G. Let $c(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices.

It is easy to see that

$$c(G) \leq \delta(G)$$

*Corresponding author: E-mail: m.abbaspour@iaukhoy.ac.ir

where G is a finite group. In this paper we determine $\delta(G)$ for some p-groups with $|G'| = p$, also we give a sharp lower bound for $\delta(G)$ in the case $Z(G) = G' \times A$ for some $A \leqslant Z(G)$ (Theorem 2.1.(c)).

Now, let G be a finite p-group and $|G'| = p$. By [[1], Theorem 3.1] G is isoclinic to a Camina p-group H of class 2 and we have

$$\delta(H) = p^m \delta(Z(H)) = p^m \sum_{i=1}^{r} m_i$$

when $p > 2$ and $Z(G) \cong \prod_{i=1}^{r} C_{m_i}$ (See [2] and [3], Theorem 2.11).

We need some known results. Moreover let $H_G = \bigcap_{g \in G} H^g$ be the core of $H \leqslant G$.

Lemma 1.1. ([[4], *Theorem* 2.2]) *Let G be a finite group. Then*

$$\delta(G) = \min\Big\{ \sum_{i=1}^{n} |G : H_i| : H_i \leqslant G \text{ for } i = 1, 2, \ldots, n \text{ and } \bigcap_{i=1}^{n} (H_i)_G = 1 \Big\}.$$

Lemma 1.2. ([[5], *Proposition* 1.5]) *Let G be a p-group such that $|G'| = p$ and $d(G) = 2$. Then G is minimal non-abelian.*

Lemma 1.3. ([[4], *Theorem* 4.12] *and* [6]) *Let G be a finite p-group of class 2 and let $Z(G)$ be cyclic. Then*

$$\delta(G) = |G : Z(G)|^{1/2} |Z(G)|$$

if G has no Q_8 section, otherwise $\delta(G) = 2|G : Z(G)|^{1/2} |Z(G)|$.

Lemma 1.4. ([7]) *Let G be a minimal non-abelian p-group. Then*

$$\delta(G) = |G : Z(G)|^{1/2} |\delta(Z(G))|,$$

unless $G \cong Q_8$ that we have $\delta(Q_8) = 8$.

2 Main Results

In this section we give our main result followed by two examples.

Let G be a finite group. Let $\mathcal{C}_i$ for $0 \leq i \leq r$ be the Galois conjugacy classes of irreducible complex characters of the group G over the rational field $\mathbb{Q}$. For $0 \leq i \leq r$, suppose that ψ_i is a representative of the class $\mathcal{C}_i$ with $\psi_0 = 1_G$. Write $\Psi_i = \sum \mathcal{C}_i$. The characters Ψ_i are called the Galois sums of G and Ψ_0 is the principal Galois sum.

Theorem 2.1. *Let G be a p-group such that $|G'| = p$. Then*

a) If $Z(G)$ is cyclic and G has no Q_8 section, then $\delta(G) = |G : Z(G)|^{1/2} |Z(G)|$, otherwise $\delta(G) = 2|G : Z(G)|^{1/2} |Z(G)|$.

b) If $d(G) = 2$, then $\delta(G) = |G : Z(G)|^{1/2} |\delta(Z(G))|$, unless $G \cong Q_8$ that we have $\delta(Q_8) = 8$.

c) If $Z(G) = G' \times A$ for some $A < Z(G)$ and for each non-principal linear character χ of A and each linear character ψ of G with $\psi_A = \chi$ holds $\Psi = |G : Z(G)|^{1/2} X$ on A (Ψ and X are the Galois sums corresponding to ψ and χ), then

$$|G : Z(G)|^{1/2}\, c(Z(G)) \leq c(G).$$

Proof. Since G is nilpotent, so $[G', G] < G'$. Therefore we have $G' \leqslant Z(G)$ and G has nilpotency class 2. Note that for each $x, y \in G; [x^p, y] = [x, y]^p = 1$, thus $G^p \leqslant Z(G)$ and $\Phi(G) = G'G^p \leqslant Z(G)$. Therefore $G/Z(G)$ is elementary abelian. By [[8], Theorem 7.5 and Example 7.6] we have $[G : Z(G)] = p^{2m}$ for some positive integer m, $\mathrm{cd}(G) = \{1, p^m\}$, and G has $|G|/p$ characters of degree 1 and $|Z(G)| - |Z(G)|/p$ irreducible characters of degree p^m. For (a) see Lemma 1.3. For (b) see Lemmas 1.2 and 1.4. Finally for (c) see [[9], Lemma 2.3] and note that $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. □

In the next Example we use Theorem 2.1.(c).

Example 2.2. *Let*

$$G =< x, y, z \,|\, x^{2^{m-2}} = 1, y^2 = z^2, xy = yx, x^z = xy, yz = zy > \ (m \geq 4).$$

Set $A =< x^2 >$. Then since $G' =< y >$, so $Z(G) = G' \times A$. It is straightforward to see that for each non-principal linear character χ of A and each linear character ψ of G with $\psi_A = \chi$ we have $\Psi = |G : Z(G)|^{1/2} X$ on A. Now, since $|G| = 2^m$ and $c(Z(G)) = 2^{m-3} + 2$, so $2^{m-2} + 4 \leq c(G)$ by Theorem 2.1.

Let $H =< x >$ and $K = \{1, y, z, yz\}$. Then $H_G \cap K_G = 1$. Thus

$$c(G) \leq \delta(G) \leq |G : H| + |G : K| = 2^{m-2} + 4$$

by Lemma 1.1 and $\delta(G) = 2^{m-2} + 4$. Note that in this example we have the equality

$$|G : Z(G)|^{1/2}\, c(Z(G)) = c(G).$$

Example 2.3. ([[*10*], *Section* 4]) *Let G be a non-abelian group of order p^4 and $Z(G)$ be an elementary abelian group. Also let $\Phi(G) < Z(G)$. Then $d(G) = 3, |G'| = p$ and $Z(G) = G' \times A$ for some $A < Z(G)$ of order p. Now, since $c(G) = p^2 + p$ and $|G : Z(G)|^{1/2}\, c(Z(G)) = p\,(p + p) = 2p^2$, so we have*

$$|G : Z(G)|^{1/2}\, c(Z(G)) > c(G).$$

3 Conclusions

In Theorem 2.1 we determined $\delta(G)$ for some p-groups G in which $|G'| = p$. Example 2.2 shows that the bound provided in Theorem 2.1.(c) is best possible. Infact the last equality of this Example is true for our p-groups G with $Z(G) = G'$ of order p (See [[2], Lemma 2.2]), however, the Example 2.3 shows that this is not the case in general.

Competing Interests

The author declares that no competing interests exist.

References

[1] Fernandez-Alcober GA, Moreto A. Groups with two extreme character degrees and their normal subgroups. Tran. of the American Math. Soc. 2001;353(6):2171-2192.

[2] Abbaspour MH, Behravesh H. A note on quasi-permutation representations of finite groups. Int. J. Contemp. Math. Sciences. 2009;27(4):1315-1320.

[3] Behravesh H. The minimal degree of a faithful quasi-permutation representation of an abelian group. Glasgow Math. J. 1997;39:51-57.

[4] Behravesh H. Quasi-permutation representations of p-groups of class 2. J. London Math. Soc. 1997;55(2):251-260.

[5] Janko Z. Finite 2-groups with no normal elementary abelian subgroups of order 8. Journal of Algebra. 2001;246:951-961.

[6] Behravesh H. Quasi-permutation representations of 2-groups of class 2 with cyclic center. LMS Lecture Note. 2001;44-49.

[7] Abbaspour MH, Behravesh H, Ghaffarzadeh G. Embeddings of minimal non-abelian p-groups. Applied Mathematics and Computation. 2011;218:658-661.

[8] Huppert B. Character theory of finite groups. de Gruyter, Berlin; 1998.

[9] Abbaspour MH, Behravesh H. The minimal degree of faithful quasi-permutation representations of minimal non-abelian p-groups. In: Proceedings of Group Theory Conference, Ferdowsi University of Mashhad. 2010;75-77.

[10] Behravesh H, Mousavi H. A note on p-groups of order $\leq p^4$. Proc. Indian Acad. Sci. (Math. Sci.). 2009;119(2):137-143.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/11711