British Journal of Mathematics & Computer Science
12(2): 1-8, 2016, Article no.BJMCS.21852
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Linear Metrics and Effective Separating Sequences

Zvonko Iljazović[1*], Anita Kuruc[1] and Lucija Validžić[1]

[1]*Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia.*

Article Information

DOI: 10.9734/BJMCS/2016/21852

Editor(s):
(1) Dariusz Jacek Jakbczak, Chair of Computer Science and Management in this Department, Technical University of Koszalin, Poland.
(2) Paul Bracken, Department of Mathematics, The University of Texas-Pan American, Edinburg, TX 78539, USA.

Reviewers:
(1) Ali Mutlu, Celal Bayar University, Turkey.
(2) Choonkil Park, Hanyang University, Republic of Korea.

Complete Peer review History: http://sciencedomain.org/review-history/11709

Original Research Article

Received: 05 September 2015
Accepted: 23 September 2015
Published: 07 October 2015

Abstract

On the real line equipped with the Euclidean metric every two effective separating sequences which have a common computable point are equivalent. We prove that the same result holds for every linear metric on the real line.

Keywords: Effective separating sequence; computable metric space; linear metric.

2010 Mathematics Subject Classification: 03D78.

1 Introduction

A dense sequence $\alpha = (\alpha_i)$ in a metric space (X, d) is said to be effective separating if we can effectively compute the distance $d(\alpha_i, \alpha_j)$ for all $i, j \in \mathbf{N}$. Two effective separating sequences α and β in (X, d) are equivalent if α can be computed from β in certain sense and conversely, if β can be computed from α. Furthermore, we say that α and β are equivalent up to isometry if there exists an isometry $f : X \to X$ such that the α and $f \circ \beta$ are equivalent.

In general, effective separating sequences in a metric space (X, d) need not be equivalent, moreover they need not be equivalent up to isometry. However, it is known that in the metric space $(\mathbf{R}, d)$,

**Corresponding author: E-mail: zilj@math.hr*

where d is the Euclidean metric on $\mathbf{R}$, every two effective separating sequences are equivalent up to isometry (although not necessarily equivalent), see [1, 2]. From this result can be concluded the following: if two effective separating sequences in $(\mathbf{R}, d)$ have a common computable point, then they are equivalent.

In this paper we generalize the previous fact by taking d to be any linear metric on $\mathbf{R}$. By a linear metric d on $\mathbf{R}$ we mean a metric such that for any real numbers $x \leq y \leq z$ we have $d(x, z) = d(x, y) + d(y, z)$.

2 Effective Separating Sequences

Let $k \in \mathbf{N} \setminus \{0\}$. A function $F : \mathbf{N}^k \to \mathbf{Q}$ is called **computable** if there exist computable (recursive) functions $a, b, c : \mathbf{N}^k \to \mathbf{N}$ such that

$$F(x) = (-1)^{c(x)} \frac{a(x)}{b(x) + 1}$$

for each $x \in \mathbf{N}^k$. A function $f : \mathbf{N}^k \to \mathbf{R}$ is called **computable** if there exists a computable function $F : \mathbf{N}^{k+1} \to \mathbf{Q}$ such that

$$|f(x) - F(x, i)| < 2^{-i}$$

for all $x \in \mathbf{N}^k$ and $i \in \mathbf{N}$ [3, 4].

A number $x \in \mathbf{R}$ is said to be **computable** if there exists a computable function $g : \mathbf{N} \to \mathbf{Q}$ such that

$$|x - g(i)| < 2^{-i}$$

for each $i \in \mathbf{N}$.

In the following proposition we state some basic facts about computable functions $\mathbf{N}^k \to \mathbf{R}$.

Proposition 2.1. *(i) If $f, g : \mathbf{N}^k \to \mathbf{R}$ are computable, then $f + g, f - g, |f| : \mathbf{N}^k \to \mathbf{R}$ are computable.*

(ii) If $F : \mathbf{N}^{k+1} \to \mathbf{R}$ is a computable function and $f : \mathbf{N}^k \to \mathbf{R}$ such that $|f(x) - F(x, i)| < 2^{-i}$ for all $x \in \mathbf{N}^k$ and $i \in \mathbf{N}$, then f is computable.

(iii) If $f, g : \mathbf{N}^k \to \mathbf{R}$ is a computable function, then the set $\{x \in \mathbf{N}^k \mid f(x) > g(x)\}$ is c.e.

Let (X, d) be a metric space and let $\alpha = (\alpha_i)$ be a sequence in X. We say that α is an **effective separating sequence** in (X, d) if the set $\{\alpha_i \mid i \in \mathbf{N}\}$ is dense in (X, d) and the function $\mathbf{N}^2 \to \mathbf{R}$,

$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$

is computable [5]. If α is an effective separating sequence in (X, d), then the triple (X, d, α) is called a **computable metric space**.

If (X, d, α) is a computable metric space and (x_i) a sequence in X, then (x_i) is said to be a **computable sequence** in (X, d, α) if there exists a computable function $F : \mathbf{N}^2 \to \mathbf{N}$ such that

$$d(x_i, \alpha_{F(i,k)}) < 2^{-k}$$

for all $i, k \in \mathbf{N}$. A point $x \in X$ is said to be **computable** in (X, d, α) if there exists a computable function $f : \mathbf{N} \to \mathbf{N}$ such that $d(x, \alpha_{f(k)}) < 2^{-k}$ for each $k \in \mathbf{N}$. Note that x is a computable point in (X, d, α) if and only if the constant sequence $(x, x, x, \dots)$ is computable in (X, d, α).

Proposition 2.2. *Let (X, d, α) be a computable metric space and let x_0 be a computable point in this space. Then the function $\mathbf{N} \to \mathbf{R}$, $i \mapsto d(x_0, \alpha_i)$ is computable.*

Proof. Let $f : \mathbf{N} \to \mathbf{N}$ be a computable function such that $d(x_0, \alpha_{f(k)}) < 2^{-k}$ for each $k \in \mathbf{N}$. In general, for all $a, b, c \in X$ we have $|d(a,c) - d(b,c)| \leq d(a,b)$, so $|d(x_0, \alpha_i) - d(\alpha_{f(k)}, \alpha_i)| \leq d(x_0, \alpha_{f(k)})$ and therefore

$$|d(x_0, \alpha_i) - d(\alpha_{f(k)}, \alpha_i)| < 2^{-k}$$

for all $i, k \in \mathbf{N}$. Now the claim follows from Proposition 2.1(ii). □

If (X, d, α) is a computable metric space, then we denote by $\mathcal{S}_\alpha$ the set of all sequences which are computable in (X, d, α).

Let (X, d) be a metric space and let α and β be effective separating sequences in this space. We say that α and β are **equivalent** and write $\alpha \sim \beta$ if β is a computable sequence in (X, d, α). It is not hard to check that effective separating sequences α and β in (X, d) are equivalent if and only if $\mathcal{S}_\alpha = \mathcal{S}_\beta$ [1]. It follows that in this case the computable metric spaces (X, d, α) and (X, d, β) have the same computable points.

Example 2.1. *Let d be the Euclidean metric on $\mathbf{R}$. Let $\alpha : \mathbf{N} \to \mathbf{R}$ be a computable function whose range is dense in $(\mathbf{R}, d)$. Then α is an effective separating sequence in $(\mathbf{R}, d)$, which follows easily from Proposition 2.1(i).*

Let (x_i) be a sequence in $\mathbf{R}$ and $a \in \mathbf{R}$. Then it is easy to prove that (x_i) is a computable sequence in $(\mathbf{R}, d, \alpha)$ if and only if (x_i) is a computable sequence in $\mathbf{R}$ (i.e. computable as a function $\mathbf{N} \to \mathbf{R}$) and a is a computable point in $(\mathbf{R}, d, \alpha)$ if and only if a is a computable number [1]. Hence $\mathcal{S}_\alpha$ is the set of all sequences in $\mathbf{R}$ which are computable (as functions $\mathbf{N} \to \mathbf{R}$).

Suppose d is the Euclidean metric on $\mathbf{R}$ and α and β are effective separating sequences in $(\mathbf{R}, d)$. Suppose α is a computable sequence in $\mathbf{R}$.

If $\alpha \sim \beta$, then by Example 2.1 β is a computable sequence in $\mathbf{R}$. Conversely, if β is a computable sequence in $\mathbf{R}$, then by Example 2.1 we have $\mathcal{S}_\alpha = \mathcal{S}_\beta$ and consequently $\alpha \sim \beta$. Hence $\alpha \sim \beta$ if and only if β is a computable sequence in $\mathbf{R}$.

Let (X, d) be a metric space, α an effective separating sequence in (X, d) and $f : X \to X$ an isometry (i.e. a surjective function such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$). Then $f \circ \alpha$, i.e. $(f(\alpha_i))_{i \in \mathbf{N}}$ is clearly an effective separating sequence in (X, d). Note the following: if x_0 is computable in (X, d, α), then $f(x_0)$ is computable in $(X, d, f \circ \alpha)$.

Example 2.2. *Let d be the Euclidean metric on $\mathbf{R}$, let $\alpha : \mathbf{N} \to \mathbf{R}$ be a computable function whose range is dense in $(\mathbf{R}, d)$ and let $c \in \mathbf{R}$. We define $\beta : \mathbf{N} \to \mathbf{R}$ by $\beta_i = \alpha_i + c$, $i \in \mathbf{N}$. Then $\beta = f \circ \alpha$, where $f : \mathbf{R} \to \mathbf{R}$ is the isometry given by $f(x) = x + c$.*

Suppose c is not a computable number. Then β is not a computable sequence in $\mathbf{R}$. Namely, if (x_i) is a computable sequence in $\mathbf{R}$, then x_i is clearly a computable number for each $i \in \mathbf{N}$. Therefore, if β were computable, β_0 would be a computable number, i.e. $\alpha_0 + c$ would be a computable number and this would imply that c is computable (in general, the difference of two computable numbers is a computable number).

Since β is not a computable sequence in $\mathbf{R}$, we have that α and β are not equivalent.

Let α and β be effective separating sequences in a metric space (X, d). We say that α and β are **equivalent up to isometry** if there exists an isometry $f : X \to X$ such that $\alpha \sim f \circ \beta$.

Example 2.2 shows that two effective separating sequences in $(\mathbf{R}, d)$, where d is the Euclidean metric on $\mathbf{R}$, need not be equivalent. However, every two effective separating sequences in $(\mathbf{R}, d)$ are equivalent up to isometry, see [1, 2].

3 Effective Separating Sequences with a Common Computable Point

There are metric spaces in which every two effective separating sequences are equivalent, see [1]. As we saw, $\mathbf{R}$ with the Euclidean metric is not such a space, but the fact that every two effective separating sequences in this metric space are equivalent up to isometry has the following consequence.

Proposition 3.1. *Let d be the Euclidean metric on $\mathbf{R}$. Suppose α and β are effective separating sequences in $(\mathbf{R}, d)$ such that the computable metric spaces $(\mathbf{R}, d, \alpha)$ and $(\mathbf{R}, d, \beta)$ have a common computable point. Then $\alpha \sim \beta$.*

Proof. Let x_0 be a common computable point of $(\mathbf{R}, d, \alpha)$ and $(\mathbf{R}, d, \beta)$.

Let us first assume that $x_0 = 0$ and that β is a computable sequence in $\mathbf{R}$.

There exists an isometry $f : \mathbf{R} \to \mathbf{R}$ such that $\alpha \sim f \circ \beta$. Since 0 is a computable point in (X, d, β), $f(0)$ is a computable point in $(X, d, f \circ \beta)$. Therefore $f(0)$ is a computable point in (X, d, α).

In general, if y and z are computable points in (X, d, α), then $d(y, z)$ is a computable number. Therefore, $d(f(0), 0)$ is a computable number.

Since f is an isometry, there exists $l \in \mathbf{R}$ such that $f(x) = x + l$ for each $x \in \mathbf{R}$ or $f(x) = -x + l$ for each $x \in \mathbf{R}$. In either case $f(0) = l$ and therefore $d(f(0), 0) = |l|$. Hence l is a computable number. It follows from Proposition 2.1(i) that the sequences $(\beta_i + l)$ and $(-\beta_i + l)$ are computable in $\mathbf{R}$.

We have $(\alpha_i) \sim (\beta_i + l)$ or $(\alpha_i) \sim (-\beta_i + l)$. Therefore α is computable sequence in $\mathbf{R}$ and it follows $\alpha \sim \beta$.

In general, let $g : \mathbf{R} \to \mathbf{R}$ be the function defined by $g(x) = x - x_0$. Then g is an isometry and we have that $g(x_0)$ is a computable point in $(X, d, g \circ \alpha)$ and in $(X, d, g \circ \beta)$. Let $\gamma : \mathbf{N} \to \mathbf{R}$ be some computable function whose range is dense in $(\mathbf{R}, d)$ (for example, we can take any computable surjection $\mathbf{N} \to \mathbf{Q}$). Then 0 is a computable point in (X, d, γ) and by the first case we have $g \circ \alpha \sim \gamma$ and $g \circ \beta \sim \gamma$. It follows $g \circ \alpha \sim g \circ \beta$. Therefore $g^{-1} \circ (g \circ \alpha) \sim g^{-1} \circ (g \circ \beta)$, hence $\alpha \sim \beta$. □

Let d be a metric on $\mathbf{R}$. We say that d is a **linear metric** if for all $x, y, z \in \mathbf{R}$ such that $x \leq y \leq z$ we have $d(x, z) = d(x, y) + d(y, z)$.

Clearly, the Euclidean metric on $\mathbf{R}$ is a linear metric. However, it is not the only linear metric as the following proposition implies.

Proposition 3.2. *Let $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a function. Then d is a linear metric if and only if there exists a strictly increasing function $f : \mathbf{R} \to \mathbf{R}$ such that*

$$d(x, y) = |f(x) - f(y)| \tag{3.1}$$

for all $x, y \in \mathbf{R}$.

Proof. Suppose d is a linear metric. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} d(0, x), & x \geq 0 \\ -d(0, x), & x \leq 0. \end{cases}$$

Let $x, y \in \mathbf{R}$, $x < y$. We claim that $f(x) < f(y)$.

If $0 \leq x < y$, then $d(0,y) = d(0,x) + d(x,y)$ and it follows $f(x) < f(y)$. In a similar way we conclude $f(x) < f(y)$ if $x < y \leq 0$. If $x < 0 < y$, then $f(x) < 0 < f(y)$.

Hence the function f is strictly increasing. We claim that (3.1) holds. It is enough to prove $d(x,y) = f(y) - f(x)$ for all $x, y \in \mathbf{R}$ such that $x < y$. However this follows easily from definition of f and the fact that d is a linear metric.

Conversely, if $f : \mathbf{R} \to \mathbf{R}$ is a strictly increasing function such that (3.1) holds, then it is easy to see that d is a linear metric. □

For example, the function $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by $d(x,y) = |\arctan x - \arctan y|$ is a linear metric. Note that d is a bounded metric.

In general, a linear metric need not be even topologically equivalent to the Euclidean metric as the following example shows.

Example 3.1. *Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by*

$$f(x) = \begin{cases} x, & x \leq 0 \\ x+1, & x > 0. \end{cases}$$

Clearly, f is strictly increasing. Let $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be defined by $d(x,y) = |f(x) - f(y)|$. Then d is a linear metric. We have

$$\{x \in \mathbf{R} \mid d(x,0) < 1\} = \langle -1, 0],$$

hence $\langle -1, 0]$ is an open ball in the metric space $(\mathbf{R}, d)$ and therefore it is an open set in $(\mathbf{R}, d)$. On the other hand, the set $\langle -1, 0]$ is not open in $\mathbf{R}$ with respect to the Euclidean metric, hence d and the Euclidean metric on $\mathbf{R}$ are not topologically equivalent.

Lemma 3.2. *Let d be a linear metric on $\mathbf{R}$ and $a, b \in \mathbf{R}$, $a < b$. Let $x \in \mathbf{R}$. Then*

$$a < x \Longleftrightarrow d(x,b) < d(a,b) \text{ or } d(x,b) < d(x,a) \tag{3.2}$$

$$x < b \Longleftrightarrow d(x,a) < d(a,b) \text{ or } d(x,a) < d(x,b) \tag{3.3}$$

Proof. Suppose $a < x$. If $x \leq b$, then $d(a,b) = d(a,x) + d(x,b)$ and it follows $d(x,b) < d(a,b)$. If $b < x$, then $d(a,x) = d(a,b) + d(b,x)$ and $d(x,b) < d(x,a)$. Hence the implication $\Rightarrow$ in (3.2) holds. Conversely, suppose $d(x,b) < d(a,b)$ or $d(x,b) < d(x,a)$. If $x \leq a$, then

$$d(x,b) = d(x,a) + d(a,b)$$

implying $d(x,a) < d(x,b)$ and $d(a,b) \leq d(x,b)$ which is impossible. Hence $a < x$ and we conclude that (3.2) holds. In a similar way we get that (3.3) holds. □

Proposition 3.3. *Let d be a linear metric and let α be an effective separating sequence in $(\mathbf{R}, d)$. Suppose x_0 is a computable point in $(\mathbf{R}, d, \alpha)$. Then the sets*

$$S = \{i \in \mathbf{N} \mid x_0 < \alpha_i\} \text{ and } T = \{i \in \mathbf{N} \mid \alpha_i < x_0\}$$

are computably enumerable.

Proof. Let us choose $y_0 \in \mathbf{R}$ such that $x_0 < y_0$. Let $r = d(x_0, y_0)$. Clearly $r > 0$. Since the set $\{\alpha_i \mid i \in \mathbf{N}\}$ is dense in $(\mathbf{R}, d)$, there exists $n \in \mathbf{N}$ such that $d(\alpha_n, y_0) < r$. We have $d(\alpha_n, y_0) < d(x_0, y_0)$ and Lemma 3.2 implies that

$$x_0 < \alpha_n.$$

Let $i \in \mathbf{N}$. By Lemma 3.2 we have

$$x_0 < \alpha_i \iff d(\alpha_i, \alpha_n) < d(x_0, \alpha_n) \text{ or } d(\alpha_i, \alpha_n) < d(\alpha_i, x_0). \tag{3.4}$$

Let $f, g : \mathbf{N}^2 \to \mathbf{R}$ be the functions defined by $f(i) = d(\alpha_i, \alpha_n)$, $g(i) = d(\alpha_i, x_0)$. By Proposition 2.2 f and g are computable functions. By (3.4) we have

$$i \in S \iff f(i) < g(n) \text{ or } f(i) < g(i).$$

It follows from this and Proposition 2.1(iii) that S is the union of two c.e. sets. Hence S is c.e. In the same way we get that T is c.e. □

Lemma 3.3. *Let d be a linear metric. Let $x, a, b \in \mathbf{R}$. If $x \leq a$ and $x \leq b$ or $a \leq x$ and $b \leq x$, then*

$$|d(x, a) - d(x, b)| = d(a, b). \tag{3.5}$$

Proof. Suppose $x \leq a$ and $x \leq b$. If $a \leq b$, then $d(x, b) = d(x, a) + d(a, b)$ and (3.5) holds. The same conclusion we get if $b \leq a$. The inequalities $a \leq x$ and $b \leq x$ imply (3.5) in the same way. □

In the next theorem we show that the claim of Proposition 3.1 holds for every linear metric d.

Theorem 3.4. *Let d be a linear metric. Suppose α and β are effective separating sequences in $(\mathbf{R}, d)$ such that the computable metric spaces $(\mathbf{R}, d, \alpha)$ and $(\mathbf{R}, d, \beta)$ have a common computable point. Then $\alpha \sim \beta$.*

Proof. Let x_0 be a common computable point of $(\mathbf{R}, d, \alpha)$ and $(\mathbf{R}, d, \beta)$. Let

$$S^+ = \{i \in \mathbf{N} \mid x_0 < \alpha_i\}, \quad S^- = \{i \in \mathbf{N} \mid \alpha_i < x_0\}$$

and

$$T^+ = \{i \in \mathbf{N} \mid x_0 < \beta_i\}, \quad T^- = \{i \in \mathbf{N} \mid \beta_i < x_0\}.$$

Let $i, k \in \mathbf{N}$. We claim that there exists $j \in \mathbf{N}$ such that

$$|d(x_0, \beta_i) - d(x_0, \alpha_j)| < 2^{-k} \tag{3.6}$$

and

$$\left(j \in S^+ \text{ and } i \in T^+\right) \text{ or } \left(j \in S^- \text{ and } i \in T^-\right) \text{ or } \left(d(x_0, \beta_i) < 2^{-(k+1)} \text{ and } d(x_0, \alpha_j) < 2^{-(k+1)}\right). \tag{3.7}$$

In order to prove this, let us first assume that $x_0 < \beta_i$. Let $r = d(x_0, \beta_i)$. Since α is a dense sequence, there exists $j \in \mathbf{N}$ such that

$$d(\alpha_j, \beta_i) < \min\{2^{-k}, r\}. \tag{3.8}$$

We have $d(\alpha_j, \beta_i) < r = d(x_0, \beta_i)$ and Lemma 3.2 implies that $x_0 < \alpha_j$. So $i \in T^+$ and $j \in S^+$. On the other hand, $|d(x_0, \beta_i) - d(x_0, \alpha_j)| \leq d(\beta_i, \alpha_j)$ and $d(\beta_i, \alpha_j) < 2^{-k}$ by (3.8), so $|d(x_0, \beta_i) - d(x_0, \alpha_j)| < 2^{-k}$. Hence (3.6) and (3.7) hold.

In the same way we conclude that (3.6) and (3.7) hold if $\beta_i < x_0$.

If $x_0 = \beta_i$, then we choose $j \in \mathbf{N}$ such that $d(x_0, \alpha_j) < 2^{-(k+1)}$. Then (3.6) and (3.7) clearly hold.

Let Ω be the set of all $(i, k, j) \in \mathbf{N}^3$ such that (3.6) and (3.7) hold. Hence for all $i, k \in \mathbf{N}$ there exists $j \in \mathbf{N}$ such that $(i, k, j) \in \Omega$. By Proposition 2.2 and Proposition 2.1 the set of all $(i, k, j) \in \mathbf{N}^3$ such that (3.6) holds is c.e. Similarly, using Proposition 3.3 we conclude that the set of all $(i, k, j) \in \mathbf{N}^3$

such that (3.7) holds is c.e. Therefore Ω is c.e. as the intersection of two c.e. sets. Since for all $i, k \in \mathbf{N}$ there exists $j \in \mathbf{N}$ such that $(i, k, j) \in \Omega$, there exists a computable function $\varphi : \mathbf{N}^2 \to \mathbf{N}$ such that

$$(i, k, \varphi(i, k)) \in \Omega \tag{3.9}$$

for all $i, k \in \mathbf{N}$ (Single Valuedness Theorem).

Suppose that $(i, k, j) \in \Omega$. We claim that $d(\beta_i, \alpha_j) < 2^{-k}$. Since (3.7) holds, we have three cases.

If $j \in S^+$ and $i \in T^+$, then $x_0 < \alpha_j$ and $x_0 < \beta_i$ and it follows from Lemma 3.3 that

$$|d(x_0, \beta_i) - d(x_0, \alpha_j)| = d(\beta_i, \alpha_j).$$

Now (3.6) implies $d(\beta_i, \alpha_j) < 2^{-k}$.

The same conclusion we get if $j \in S^-$ and $i \in T^-$. If $d(x_0, \beta_i) < 2^{-(k+1)}$ and $d(x_0, \alpha_j) < 2^{-(k+1)}$, then clearly $d(\beta_i, \alpha_j) < 2^{-k}$.

Hence $(i, k, j) \in \Omega$ implies $d(\beta_i, \alpha_j) < 2^{-k}$. We conclude from (3.9) that

$$d(\beta_i, \alpha_{\varphi(i,k)}) < 2^{-k}$$

for all $i, k \in \mathbf{N}$. Hence β is a computable sequence in $(\mathbf{R}, d, \alpha)$, i.e. $\alpha \sim \beta$. □

4 Conclusion

In this paper we have examined conditions under which two effective separating sequences in a metric space are equivalent. We first focused on the metric space $(\mathbf{R}, d)$, where d is the Euclidean metric on $\mathbf{R}$. If α and β are effective separating sequences in this space, then α and β need not be equivalent. However, we saw that results from [1, 2] implied the following: if $(\mathbf{R}, d, \alpha)$ and $(\mathbf{R}, d, \beta)$ have a common computable point, then α and β are equivalent.

In the main part of the paper we have generalized this result by introducing the notion of a linear metric and by showing that the latter statement holds for any linear metric d.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Iljazović Z. Isometries and computability structures. Journal of Universal Computer Science 2010;16:2569-2596.

[2] Melnikov AG. Computably isometric spaces. Journal of Symbolic Logic 2013;78:1055-1085.

[3] Pour-El MB, Richards I. Computability in analysis and physics. Springer-Verlag, Berlin-Heidelberg-New York; 1989.

[4] Weihrauch K. Computable analysis. Springer, Berlin; 2000.

[5] Yasugi M, Mori T, Tsujji Y. Effective properties of sets and functions in metric spaces with computability structure. Theoretical Computer Science. 1999;66:127-138.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/11709