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Cell Arrangement Method for Solving Systems of Linear Equations in Three Unknown

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we develop an approach for finding the cofactor, ad joint, determinant and inverse of a three by three matrix under the Cell Arrangements method using the coefficient matrix of a given systems of linear equation in three unknowns. The method takes out completely the seemingly daunting task in evaluating such matrices associated to the standard matrix method in solving simultaneous equation in three variable. Unlike the standard matrix method that goes through a lengthy process to obtain separately all the matrices necessary for the determination of the unknowns, the structural frame of the Cell Arrangement method comes in handy and are consistent with the results from systems that have unique solutions. This alternative approach provides all the vital hybrid matrices of the coefficient matrix needed in the determination of the unknowns of the system of equations in three variables. It is our view that by far, the Cell arrangement method is easy to work with and less prone to errors that are often connected with other known methods.

Keywords: Vector product; array; cofactors; ad joint; determinant; inverse.

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1 Introduction

Simultaneous equation is a common method used in solving systems of linear equation in two unknowns. A repeated use of simultaneous equation in three or more unknowns becomes cumbersome to handle to the extent that mistake in one step may affect the entire determination of the unknown quantities. A better approach and more effective way for dealing with higher systems of linear equation is by the use of matrices and certain peculiar properties associated to them.

One of such methods was established by G. Cramer (1704-1752) [1] a Swiss mathematician, where he adopted four different determinants one from the coefficient matrix of the given linear equations and three other hybrid determinants from the same coefficients matrix of which each column in turn is replaced with the right hand side (RHS of the system. The unknown was found by forming ratios of the hybrid determinants with the determinant of the coefficient matrix. The glitch in this method is that if the coefficient matrix is singular the method fails and in practice, Cramer's rule is rarely used to solve systems of order higher than three(3) [1] The advantage of this method worth noting is the light it sheds on the behavior of simultaneous linear equation [2].

The standard matrix method which uses the ad joint, determinant and inverse properties of a matrix to determine the unknown quantities of a system is quiet laborious and requires constant practice in order to master the steps involved. Thus transition from the traditional simultaneous equation in two variables to solving three variables using matrix method is enormous and for many people who take mathematics as a pre-requite course or related programs that requires mathematics, the knowledge gap needs to be bridged.

The purpose of this paper is to introduce matrix approach of solving systems of linear equation using cell arrangements by an introduction of carefully ordered vector product.

2 Related Works

When Linear equations arise from a practical problem, the coefficients are unlikely to be small integers and the arithmetic can get heavy [3]. It is for this reason that we have opted to review the work done by earlier authors on solving systems of linear equation using matrices since it offers suitable properties which enable us to critique a given system as having unique , infinite or one with no solution.

Solving systems of linear equations by the standard method comprises four basic processes [4].

The given system is firstly, put in the matrix representation

$$AX = b \tag{1}$$

where A represents the coefficient of matrix for the system, X and b represents column vectors for the unknown variables and the constants of the RHS of the given system. For the purpose of the work at hand we shall deal with a system of linear equation in three unknowns.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{2}$$

$$\text{where } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{3}$$

This is followed by finding the determinant of the coefficient matrix which can be developed along any of the rows or any of the columns. Symbolically the determinant is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{4}$$

We shall show the case where it is developed along the first row. For instance, each element and the sign associated to the position it occupies in the first row is use to multiply the lesser order determinant form by the deletion of the column and row the particular element is located. This gives

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \tag{5}$$

The sign associated to the position an element in an array occupies, is found as the sum of the row and column number of the index to which (-1) is raised, that is $(-1)^{i+j}$ or you may determine it manually by moving in $(+)$ and $(-)$ alternation, starting from the first row and first column of the given array ([1] and [2]). If there are more zeros in a particular row or column, then it would be more instructive to find the determinant along such row or column.

Next the Minors of each element in the matrix A are found by deleting row and column of each particular element in that row and in that column and the determinant of the resulting arrays found. This would give in all a total of nine, two by two determinants namely

$$\begin{matrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{matrix} \tag{6}$$

By renaming these minors with their associated designated signs we generate the elements of the cofactors as shown below.

$$\begin{matrix} A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ A_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & A_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ A_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & A_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{matrix} \tag{7}$$

Once the cofactors of the given coefficient matrix are deduced from the signed minors they are written out as a matrix array called the cofactor matrix and it is usually denoted and defined as

$$C = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{8}$$

The adjoint matrix is obtained by finding the transposition of the matrix in $eq(8)$ which yields

$$Adj(A) = C^T = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \tag{9}$$

The last property to be pursue in our quest of using matrix approach in solving systems of linear equation in three unknown is to determine the inverse matrix A^{-1} of the matrix A. This is easily done by find the product of the reciprocal of the determinant of equation (5) [5] and the adjoint matrix of equation (9) i.e.

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \tag{10}$$

Finally using equations (10) and (3) the unknown of the system are uniquely found provided $|A|$ is not equivalent to zero in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X = \frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{11}$$

The advantage of the method is that it is structured and by extension it could be applied on higher order nonsingular matrices. The inherent lapses associated to the standard matrix method is also due to the fact that, it is structured and very laborious. A common error that may occur is the omission of the prescribed signs for the cofactors which do not actually surface in their development and if not remedied, the entire determination of the unknown would yield inaccurate results.

[6] Observe that the Calculation of the entries in the adjoint or adjugate matrix from their basic definition can seem a very daunting prospect and to overcome the none introduction of the designated signs relating the minors to the cofactors they propose an alternative approach, in a way that the original entries of the matrix in equation (3) are written repeatedly in each section of a quadrant as shown below.

$$\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \\ \hline a_{11} & a_{12} & a_{13} & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & a_{33} \end{array}$$

This is followed by the deletion of the extreme elements round the quadrant. Once that is done, all possible two by two determinants of the remaining array are evaluated producing

$$\begin{array}{c|c} a_{22}\ a_{23} & a_{21}\ a_{22} \\ a_{32}\ a_{33} & a_{31}\ a_{32} \\ \hline a_{12}\ a_{13} & a_{11}\ a_{12} \\ a_{22}\ a_{23} & a_{21}\ a_{22} \end{array}$$

the same results for the entries of the cofactors as in equations (7) and (8). Once the cofactor matrix is obtained, the adjoint, inverse matrix and the determinant are used accordingly to retrieve the unknown being sought for. Clearly, the innovation introduced by these writers is that, the computations of the cofactor matrix is simpler and less prone to errors. The approach proposed by [6]; however does not work for matrix whose order is greater than three (3).

3 Mathematical Methods

The results of a cross product of two vectors $\boldsymbol{F_1} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{F_2} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ is given by $\boldsymbol{F_1} \times \boldsymbol{F_2} = (a_2b_3 - a_3b_2)\boldsymbol{i} + (a_3b_1 - a_1b_3)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$ where the element in the p^{th} component of the cross product is obtained by omitting only the p^{th} column and evaluating the determinant of the remaining components in an anticlockwise cyclic manner. This idea may be exploited in obtaining the cofactor matrix without associating the designated sign of the determinants of their respective minors.

Theorem

Suppose the rows of a 3×3 coefficient matrix A of a system of linear equation represents the components of the vectors $V_1 =< a_{21}\ a_{22}\ a_{23}\ >, V_2 = < a_{31}\ a_{32}\ a_{33}\ >$ and $V_3 = < a_{11}\ a_{12}\ a_{13}\ >$ then the

i. cross products $V_1 \times V_2$; $V_2 \times V_3$; $V_3 \times V_1$ generates the row entries of the cofactor matrix without the placed sign of the minors of the original matrix

ii. scalar triple products $V_3.(V_1 \times V_2) = |A|$

Proof:

Let the entries of the coefficient matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ of a given system of linear equations which is consistent be defined by the vectors

$V_1 =< a_{21};\ a_{22}\ ;\ a_{23}\ >, V_2 =< a_{31};\ a_{32};\ a_{33}\ >, V_3\ =< a_{11};\ a_{12};\ a_{13}\ >$

then

$$\begin{aligned} V_1 \times V_2 &= < a_{22}a_{33} - a_{23}a_{32}\ ;\ a_{23}a_{31} - a_{21}a_{33}\ ;\ a_{21}a_{32} - a_{22}a_{31} > \\ &= < \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} ; \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} ; \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} > \text{No place signed} \\ &= < A_{11}\ ;\ A_{12}\ ;\ A_{12} > \end{aligned} \qquad (a)$$

also

$$\begin{aligned} V_2 \times V_3 &= < a_{32}a_{13} - a_{33}a_{12}\ ;\ a_{33}a_{11} - a_{31}a_{13}\ ;\ a_{31}a_{12} - a_{32}a_{11} > \\ &= < \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} ; \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix} ; \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} > \text{No place sign} \end{aligned}$$

$$= < A_{21}\ ;\ A_{22}\ ;\ A_{23} > \qquad (b)$$

Similarly

$$\begin{aligned} V_3 \times V_1 &= < a_{12}a_{23} - a_{13}a_{22}\ ;\ a_{13}a_{21} - a_{11}a_{23}\ ;\ a_{11}a_{22} - a_{12}a_{21} > \\ &= < \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} ; \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} ; \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > \text{Having no place sign} \\ &= < A_{31}\ ;\ A_{32}\ ;\ A_{33} > \end{aligned} \qquad (c)$$

Finally writing out the results of each of these cross products in equations (a) (b) and (c) as the row entries of a 3×3 matrix, the cofactor matrix of the original matrix is determined.

ii. $V_3.(V_1 \times V_2) = < a_{11};\ a_{12};\ a_{13}\ >.< A_{11}\ ;\ A_{12}\ ;\ A_{13} >$
$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}\ = |A|$

The Product $V_3.(V_1 \times V_2)$ is known in vector Analysis as the scalar triple product. This evaluate a single unique real number associated to the matrix called the determinant of the coefficient matrix. The determinant is important since geometrically, it's absolute value represents the volume of the parallelepiped spanned by the vectors V_1, V_2 and V_3.

By carefully arranging the rows of a 3×3 matrix in three different cells in pairs, starting with the second row and repeating the last row of a pair in the next cell, the co- factor matrix , the adjoint matrix the determinant are easily obtained and hence the inverse of the matrix under consideration found at the same time. A prototype of this approach is shown using the matrix

$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ and a demonstration of the method is illustrated with an example.

$$\begin{array}{ccc} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ \hline c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ \hline a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{array}$$

$$C = \begin{bmatrix} (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1) \\ (c_2a_3 - c_3a_2) & (c_3a_1 - c_1a_3) & (c_1a_2 - c_2a_1) \\ (a_2b_3 - a_3b_2) & (a_3b_1 - a_1b_3) & (a_1b_2 - a_2c_1) \end{bmatrix}$$

$$C^T = adj(A) = (b_1c_2 - b_2c_1) \quad (c_1a_2 - c_2a_1) \quad (a_1b_2 - a_2c_1)$$

$$|A| = a_1(b_2c_3 - b_3c_2) + a_2\,(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

Clearly, it can be seen that all the rows of the cofactor matrix give the precise definition of a cross product of the element of the original matrix arranged in pairs following this approach. A Transposition of the cofactor matrix gives the adjoint matrix A^* of matrix A. Two other interesting properties of the matrix A that can be derived from the above is the determinant $|A|$ and the inverse A^{-1} of matrix A. The determinant can be shown to be the term by term multiplication of the first row of the last cell and the first column of the ad joint matrix and this is shown in the layout by the arrows, (i.e. the scalar product along the row and column specified) while the inverse matrix A^{-1} is easily obtained by the scalar multiplication of the reciprocal of the determinant and the adjoint matrix.

4 Results and Discussion

An immediate application is solving systems of linear equation in three unknowns. We illustrate the Cell arrangement method with a system having the following information.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{array}{ccc} 2 & 1 & 1 \\ 3 & 1 & -2 \\ \hline 3 & 1 & -2 \\ 1 & 2 & 3 \\ \hline 1 & 2 & 3 \\ 2 & 1 & 1 \end{array}$$

$$C = \begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}$$

$$C^T = \begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}$$

$$|A| = 1(-3) + 2(7) + 3(-1) = 8$$

$$\therefore\ A^{-1} = \frac{1}{8}\begin{pmatrix} 0 & 2 & -1 \\ -7 & 2 & 8 \\ -14 & 5 & 13 \end{pmatrix}$$

$$hence\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -3 & 7 & -1 \\ 7 & -11 & 5 \\ -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

The cofactor ad joint procedure for solving linear equations is, rather tedious, especially when the order is much higher, the arithmetic becomes quite challenging. To save labor and to greatly facilitate the solution of the system [7], there is therefore the need to seek for an alternative approach without compromising the underlining principle of the matrix method. It is in this light that the Cell arrangement method becomes an indispensable tool in the determination of the cofactors, adjugate matrix, the determinant, the inverse and hence the unknown quantities of the system of equations. The advantage of the Cell arrangement method over the standard matrix approach is that, the steps involve in obtaining the properties of the coefficient matrix necessary for the determination of the unknown are less laborious and less time consuming. This is so since the same procedure is repeated three times on each paired cells and only a j^{th} column is deleted and also ensuring that the cofactor to occupy that position is evaluated in an anticlockwise manner. The method really works faster especially when the arithmetic of the procedure discussed is done mentally without having to write out the determinants that evaluates each cofactor and more so the necessity of assignment of the designated sign in the computation of the cofactors involved is completely eliminated. In contrast to the standard methods, much effort and time is spent on the determination of the cofactors in each particular position by the deletion of both the i^{th} and the j^{th} entries of the original matrix A and the determinant of the remaining array found, multiplied by the scalar $(-1)^{i+j}$ of that position.

The only inherent setback for the Cell arrangement method is that it works only for linear equation in three variables and the process of finding the cross product of the respective row vectors may pose a challenge since the ordering of the row vectors are extremely important to our search for the solution. This method permits defined ordering of the vectors we generate from the coefficient matrix. This is so because of the manner in which the entries of the cofactor matrix are churned out. They follow precisely the definition of a cross product of two vectors which are strictly defined for three-dimensional vectors [8].

5 Conclusion

An alternative approach that provides all the vital properties of a coefficient matrix needed in getting the unknown of a system of equations is introduced. It is our view that the Cell arrangement method is easy to work with and less prone to errors as compared to the standard matrix method which is structured and the processes involving their usage can seem a very daunting prospect.

Competing Interests

Authors have declared that no competing interests exist.

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