Journal of Advances in Mathematics and Computer Science

35(8): 65-75, 2020; Article no.JAMCS.63334

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Generalized Hadamard Matrices and Generalized Hadamard Graphs

W. V. Nishadi[1*] and A. A. I. Perera[1]

[1] *Department of Mathematics, Faculty of Science, University of Peradeniya, Peradeniya, Sri Lanka.*

Authors' contributions

This work was carried out in collaboration between both authors. Author WVN designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author AAIP managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2020/v35i830314

Editor(s):

(1) Dr. Mohd Zuki Salleh, Universiti Malaysia Pahang, Malaysia.

Reviewers:

(1) Emad Ahmad Az-Zobi, Mutah University, Jordan.

(2) Rasajit Kumar Bera, National Institute of Technical Teachers Training & Research (NITTTR), India.

Complete Peer review History: http://www.sdiarticle4.com/review-history/63334

***Received:** 24 September 2020*

***Accepted:** 29 November 2020*

***Published:** 03 December 2020*

Original Research Article

Abstract

Hadamard matrices and their applications have steadily and rapidly grown during the last 2 decades. Due to that many researchers have developed various concepts on Hadamard matrices. This paper concentrates on Generalized Hadamarad matrices. In the first part of this work, some new results on construction of generalized Hadamard matrices $GH(p, p^n)$ over C_p are introduced. In the second part, graphs obtained from generalized Hadamard matrices are introduced, namely generalized Hadamard graphs. In particular, we show that the generalized Hadamard graphs are p^n- regular. Our results have been illustrated by constructing p^n- regular graphs for different values of p and n.

Keywords: Generalized Hadamard matrix; Hadamard matrix; Kronecker product; Latin square; regular graph.

2010 Mathematics Subject Classification: 53C25, 83C05, 57N16.

**Corresponding author: E-mail: vindinisha@hotmail.com;*

1 Introduction

Hadamard matrices and their applications have steadily and rapidly grown during the last 2 decades. Due to that many researchers have developed various concepts on Hadamard matrices. A Hadamard matrix H of order n is an $n \times n$ array with entries ± 1, which satisfies $HH^T = nI_n$, where I_n denotes the identity matrix of order n and H^T is the transpose of H. It is recognized that n has to be necessarily $1, 2$ or a multiple of 4, but there is no certainty whether such a Hadamard matrix exists at every possible order. The Hadamard Conjecture claims that there exists a Hadamard matrix of order $4t$ for every natural number t [1]; [2].

An n-Hadamard graph is a graph of $4n$ number of vertices defined in terms of a Hadamard matrix $H_n = (h_{ij})$ by constructing the vertices using $4n$ symbols r_i^+, r_i^-, c_j^+, and c_j^-, where r stands for "row" and c stands for "columns" and constructing two edges $(r_i^{\pm}, c_j^{\pm})$ for each matrix element such that $h_{ij} = 1$ and $(r_i^{\pm}, c_j^{\mp})$ for each matrix element such that $h_{ij} = -1$ [3]; [4].

This paper is concerned with the generalization of Hadamard matrices and graphs induced by generalized Hadamard matrices. Let C be a multiplicative group of order w. An $v \times v$ matrix $M = [m_{ij}]$ with entries from C where w divides v is a generalized Hadamard matrix denoted by $GH(w, v)$over C if, for all $i \neq j$, the sequence of quotients $m_{ij} m_{jk}^{-1}, 1 \leq k \leq v$, contains each element of C exactly v/w times. If M is a $GH(w, v)$ over C then $MM^* = vI_v + v/w(\sum_{u \in C} u)(J_v - I_v)$, where M^* denotes the conjugate transpose of M [5]; [6]. If C is abelian then $GH(w, v)^T$ is also a generalized Hadamard matrix, where $GH(w, v)^T$ denotes the transpose of $GH(w, v)$. This result does not generalize to non-abelian groups, as shown by Craigen and de Launey [7].

When C is changed, the definition alters accordingly. Let C_p be the cyclic group of all complex p^{th} root of unity. A square matrix $M = [m_{ij}]$ of order v over C_p is called a generalized Hadamard matrix denoted by $GH(p, v)$, if $MM^* = vI_v$, where M^* is the conjugate transpose of M and and I_v is the identity matrix of order v. Further, it contains each element of C_p exactly vp times [8].

A generalized Hadamard matrix of order 3 over C_3 is given to be:

$$\begin{bmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

Two generalized Hadamard matrices H_1 and H_2 are said to be equivalent if and only if one can be obtained from the other by permuting the rows (columns) and a series of multiplications of elements of group of rows (columns). An $GH(p, v)$ can be reduced to the standard form (or normalized form) in which the initial row and column contain only 1. Then,

$$\sum_{j=1}^{v} m_{ij} = \sum_{j=1}^{v} m_{ij}^C = 0, i = 2, 3, \cdots, v \tag{1.1}$$

$$\sum_{i=1}^{v} m_{ij} = \sum_{i=1}^{v} m_{ij}^C = 0, j = 2, 3, \cdots, v \tag{1.2}$$

A square sub matrix after omitting the first row and first column of standard form of $GH(p, v)$ is known as the core [9].

The term Kronecker product, also known as the tensor product is defined, as it is very useful in this context. If $A = [a_{ij}]$ is an $u \times v$ matrix for $i = 1, 2, \cdots, u$ and $j = 1, 2, \cdots, v$ and B is any $p \times q$ matrix then the Kronecker product of A and B, denoted by $A \otimes B$, is the $up \times vq$ matrix formed by multiplying each a_{ij} element by the entire matrix B. That is, [6]

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1v}B \\ a_{21}B & a_{22}B & \cdots & a_{2v}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{u1}B & a_{u2}B & \cdots & a_{uv}B \end{bmatrix}_{u \times v}$$

It leads to the following lemma.

Lemma 1.1. *If $M_1$is an $GH(p, v_1)$ matrix over C and M_2 is an $GH(p, v_2)$ matrix over C, then the Kronecker product $M_1 \otimes M_2$ is an $GH(p, v_1v_2)$ matrix over C [8].*

We examine the construction of generalized Hadamard matrices with the properties of Latin square. A Latin Square of order n is an $n \times n$ matrix containing n different symbols that each symbol occurs in each row and each column exactly once [10]. The following is an example of a Latin square of order 3.

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

Once the construction of generalized Hadamard matrices is accomplished, we consider the generalization of the Hadamard graphs and discuss their properties [11].

Although it is easy to represent a graph by a diagram of points joined by lines, such a representation may be irrelevant if we wish to store a large graph in a computer. One way of storing a simple graph is by listing the vertices adjacent to each vertex of the graph known as adjacency matrix [12].

Let $G = (V, E)$ be a simple graph with vertex set $V = \{1, 2, \cdots, n\}$. The adjacency matrix of G, denoted by $A(G)$ is defined as the $n \times n$ matrix $A(G) = [a_{ij}]$ where,

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent in } G \\ 0 & \text{otherwise.} \end{cases}$$

2 Generating Generalized Hadamard Matrices

The main purpose of this paper is to generalize the Hadamard matrices for order of prime powers over cyclic groups of prime orders. To construct a $GH(p, p^n)$ over C_p, we can use the following steps.

Step 1: Let the elements be $x_1, x_2, x_3, , x_p$.

Step 2: Write a Latin square by using the cyclic shifting method [13] by taking the first row $x_1, x_2, , x_p$. Convert into the standard form by multiplying inverse elements.

Step 3: Label the first row and first column elements of the core as $A_1, A_2, , A_{p-1}$. Observe that, it has a triangular pattern.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & A_1 & A_2 & A_3 & \cdots & A_{p-2} & A_{p-1} \\ 1 & A_2 & A_2A_3A_1^{-1} & A_3A_4A_1^{-1} & \cdots & A_{p-1}A_{p-2}A_1^{-1} & A_{p-1}A_1^{-1} \\ 1 & A_3 & A_3A_4A_1^{-1} & A_3A_4A_5A_1^{-1}A_2^{-1} & \cdots & A_{p-1}A_{p-2}A_1^{-1}A_2^{-1} & A_{p-1}A_2^{-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & A_{p-3} & A_{p-2}A_{p-3}A_1^{-1} & A_{p-1}A_{p-2}A_{p-3}A_1^{-1}A_2^{-1} & \cdots & A_{p-1}A_{p-2}A_{p-4}^{-1}A_{p-5}^{-1} & A_{p-1}A_{p-4}^{-1} \\ 1 & A_{p-2} & A_{p-1}A_{p-2}A_1^{-1} & A_{p-1}A_{p-2}A_1^{-1}A_2^{-1} & \cdots & A_{p-1}A_{p-2}A_{p-3}^{-1}A_{p-4}^{-1} & A_{p-1}A_{p-3}^{-1} \\ 1 & A_{p-1} & A_{p-1}A_1^{-1} & A_{p-1}A_2^{-1} & \cdots & A_{p-1}A_{p-3}^{-1} & A_{p-1}A_{p-2}^{-1} \end{bmatrix}$$

Step 4: Substitute $A_i = \omega^i$ for $i = 1, 2, , p-1$, where ω is the p^{th} root of unity.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{p-3} & \omega^{p-2} & \omega^{p-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2p-6} & \omega^{2p-4} & \omega^{p-2} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3p-9} & \omega^{2p-6} & \omega^{p-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \omega^{p-3} & \omega^{2p-6} & \omega^{3p-9} & \cdots & \omega^9 & \omega^6 & \omega^3 \\ 1 & \omega^{p-2} & \omega^{2p-4} & \omega^{2p-6} & \cdots & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^{p-1} & \omega^{p-2} & \omega^{p-3} & \cdots & \omega^3 & \omega^2 & \omega \end{bmatrix} \tag{2.1}$$

Step 5: Use the Kronecker product to construct the normalized $GH(p, p^2)$ matrix.
Applying the Kronecker product construction repeatedly, one can construct $GH(p, p^n)$, where $n \in \mathbb{N}$.

The method described above is proved by the conditions mentioned (1.1) and (1.2) for the standard form of a generalized Hadamard.

Theorem 2.1. *Let $p > 2$ be a prime number. Then there exists $GH(p, p)$ over C_p with the condition $h_{ij} = \omega^{ij(modp)}$.*

Proof. The core elements in (2.1) are represented by $h_{ij} = \omega^{ij(modp)}$.
Now consider

$$\begin{aligned} \sum_{i=1}^{p} m_{ij} &= m_{1j} + \sum_{i=2}^{p} m_{ij} \\ &= 1 + \sum_{i=1}^{p-1} h_{ij} \\ &= 1 + h_j + h_{2j} + \cdots + h_{(p-1)j} \\ &= 1 + \omega^{j(modp)} + \omega^{2j(modp)} + \cdots + \omega^{(p-1)j(modp)} \\ &= 1 + \omega^{j(modp)} \left[\frac{1 - \omega^{(p-1)j(modp)}}{1 - \omega^{j(modp)}} \right]; \quad \text{since} \quad j < p, \omega^{j(modp)} \neq 1 \\ &= 1 + \left[\frac{\omega^{j(modp)} - \omega^{pj(modp)}}{1 - \omega^{j(modp)}} \right] \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

The similar proof can be used to show that $\sum_{i=1}^{p} m_{ij}^C = 0$ and equation (1.1) considering any column. Thus, we can say that there exists a generalized Hadamard matrix $GH(p, p)$. □

2.1 Case I: $p = 3, n = 1$

In these subsections, we construct the Generalized Hadamard matrices for prime orders, and prime power orders.

In fact, the generalized Hadamard matrix of order 3 is to be constructed.The matrix with coefficients $A_i's$ is of the form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & A_1 & A_2 \\ 1 & A_2 & A_2 A_1^{-1} \end{bmatrix}$$

After Substituting $A_i = \omega^i$ for $i = 1, 2$, where ω is the 3^{rd} root of unity we will get the matrix of the form:

$$H_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \tag{2.2}$$

To verify the obtained matrix is a generalized Hadamard matrix, consider the product $H_1 H_1^*$:

$$\begin{aligned} H_1 H_1^* &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}^* \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 3 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega+\omega^2 & 3 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= 3I_3 \end{aligned}$$

Thus, H_1 is a generalized Hadamard matrix.

2.2 Case II: $p = 5, n = 1$

The generalized Hadamard matrix of order 5 over C_5 is considered. Here ω is the 5^{th} root of unity. That is $\omega^5 = 1$. This implies $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

$$H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \tag{2.3}$$

2.3 Case III: $p = 7, n = 1$

The generalized Hadamard matrix of order 7 over C_7 is considered. Here ω is the 7^{th} root of unity. That is $\omega^7 = 1$. This implies $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$.

$$H_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \\ 1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \tag{2.4}$$

2.4 Case IV: $p = 3, n = 2$

Here, we use the lemma 1.1 which consists of the Kronecker product on generalized Hadamard matrices. If H_1 is $GH(3,3)$ and H_2 is $GH(3,3)$, then there exists $GH(3,9)$ over C_3.

$$H_4 = H_1 \otimes H_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 & \omega \\ 1 & 1 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega^2 & \omega & 1 \\ 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega & \omega^2 & \omega^2 & 1 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega & 1 & \omega^2 \end{bmatrix} \qquad (2.5)$$

The normalized generalized Hadamard matrices of orders less than 100 which can be constructed using recursive algorithm are tabulated as follows. (Table 1)

Table 1: $GH(p, p^n)$ matrices of orders up to 100

n-values	0	10	20	30	40	50	60	70	80	90
0										
1		11		31	41		61	71	3	
2	2			2						
3	3	13	23		43	53		73	83	
4	2						2			
5	5		5							
6		2								
7	7	17	3	37	47		67			97
8	2									
9	3	19	29		7	59		79	89	

3 Generating Generalized Hadamard Graphs

It is useful to have a common method of finding generalized Hadamard graphs from generalized Hadamard matrices as it was not done so far in the literature. We present such a method in this paper by using the properties of Latin squares.

Definition 3.1. Let the distinct elements of generalized Hadamard matrix $GH(p, p^n)$ as $1, \omega, \cdots, \omega^{p-i}$. Then the Latin square using cyclic shifting method with elements $\{1, \omega, \cdots, \omega^{p-i}\}$ can be converted into row matrix with elements $\{L_0, L_1, , L_{p-1}\}$ such that $L_m = [h_{im}]$.

$$A = \begin{bmatrix} 1 & \omega & \cdots & \omega^{p-1} \\ \omega & \omega^2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{p-1} & 1 & \cdots & \omega^{p-2} \end{bmatrix} \tag{3.1}$$

$$= \begin{bmatrix} L_0 & L_1 & \cdots & L_{p-1} \end{bmatrix} \tag{3.2}$$

Definition 3.2. A generalized Hadamard graph of $2p^{n+1}$ number of vertices and p^{2n+1} edges defined in terms of generalized Hadamard matrix $GH(p,p^n)$ by constructing:

i) the vertices using $2p^{n+1}$ symbols $r_i^1, r_i^\omega, \cdots, r_i^{p-1}, c_j^1, c_j^\omega, \cdots, c_j^{p-2}$ and c_j^{p-1}, where r stands for 'row' and c stands for 'column' for $i, j = 1, 2, \cdots, p^n$.

ii) and the edges $(r_i^{L_0}, c_j^{L_m})$ such that $h_{ij} = \omega^m$, where $m = 0, 1, \cdots, p-1$.

Theorem 3.1. *Let $p > 2$ be a prime number. Then there exists a p^n- regular graph from generalized Hadamard matrix $GH(p,p^n)$ over C_p.*

Proof. It is well-known that the necessary and sufficient conditions for a k- regular graph of order n exists if $n \geq k+1$ and that nk is even. Equality holds when the graph is a complete graph. In the above constructed generalized Hadamard graph has $n = 2p^{n+1}$ number of vertices and it is regular of $k = p^n$. We will show that the above two conditions are satisfied by n and k.

Claim: nk is even and $n - (k+1) > 0$.

It can be seen that nk is even as n is a multiple of 2.

Let's consider $n - (k+1)$

$$\begin{aligned} n - (k+1) &= 2p^{n+1} - (p^n + 1) \\ &= 2p^{n+1} - p^n - 1 \\ &= p^n(2p-1) - 1 > 0 \quad ; \quad \text{since} \quad p > 2 \quad \text{and} \quad n \in \mathbb{N} \end{aligned}$$

Thus, there exist a p^n- regular graph on $2p^{n+1}$ number of vertices. □

3.1 Case I: Graph obtained from $GH(3, 3^n)$

In these subsections, we construct the Generalized Hadamard graphs for prime orders and prime power orders of generalized Hadamard matrices using the Python software.

The Latin square related with $GH(3, 3^n)$ can be shown as follows:

$$A_1 = \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{bmatrix} = \begin{bmatrix} L_0 & L_1 & L_2 \end{bmatrix} \tag{3.3}$$

Define 6.3^n symbols $r_i^1, r_i^\omega, r_i^{\omega^2}, c_j^1, c_j^\omega, c_j^{\omega^2}$ for $i, j = 1, 2, \cdots, 3^{n-1}$ as the vertices of the graph. According to the definition, edges $(r_i^{L_0}, c_j^{L_m})$ if $h_{ij} = \omega^m$, where $m = 0, 1, 2$. Then we have 3^{2n+1} number of edges.

3.1.1 When $n = 1$

This gives the graph for the generalized Hadamard matrix $GH(3,3)$. We can obtain the following edges set:

$$h_{11} = 1 \Rightarrow (r_1^{L_0}, c_1^{L_0}) \Rightarrow (r_1^1, c_1^1), (r_1^{\omega}, c_1^{\omega}), (r_1^{\omega^2}, c_1^{\omega^2}) \quad (3.4)$$

$$h_{12} = 1 \Rightarrow (r_1^{L_0}, c_2^{L_0}) \Rightarrow (r_1^1, c_2^1), (r_1^{\omega}, c_2^{\omega}), (r_1^{\omega^2}, c_2^{\omega^2}) \quad (3.5)$$

$$h_{13} = 1 \Rightarrow (r_1^{L_0}, c_3^{L_0}) \Rightarrow (r_1^1, c_3^1), (r_1^{\omega}, c_3^{\omega}), (r_1^{\omega^2}, c_3^{\omega^2}) \quad (3.6)$$

$$h_{21} = 1 \Rightarrow (r_2^{L_0}, c_1^{L_0}) \Rightarrow (r_2^1, c_1^1), (r_2^{\omega}, c_1^{\omega}), (r_2^{\omega^2}, c_1^{\omega^2}) \quad (3.7)$$

$$h_{22} = \omega \Rightarrow (r_2^{L_0}, c_2^{L_1}) \Rightarrow (r_2^1, c_2^{\omega}), (r_2^{\omega}, c_2^{\omega^2}), (r_2^{\omega^2}, c_2^1) \quad (3.8)$$

$$h_{23} = \omega^2 \Rightarrow (r_2^{L_0}, c_3^{L_2}) \Rightarrow (r_2^1, c_3^{\omega^2}), (r_2^{\omega}, c_3^1), (r_2^{\omega^2}, c_3^{\omega}) \quad (3.9)$$

$$h_{31} = 1 \Rightarrow (r_3^{L_0}, c_1^{L_0}) \Rightarrow (r_3^1, c_1^1), (r_3^{\omega}, c_1^{\omega}), (r_3^{\omega^2}, c_1^{\omega^2}) \quad (3.10)$$

$$h_{32} = \omega^2 \Rightarrow (r_3^{L_0}, c_2^{L_2}) \Rightarrow (r_3^1, c_2^{\omega^2}), (r_3^{\omega}, c_2^1), (r_3^{\omega^2}, c_2^{\omega}) \quad (3.11)$$

$$h_{33} = \omega \Rightarrow (r_3^{L_0}, c_3^{L_1}) \Rightarrow (r_3^1, c_3^{\omega}), (r_3^{\omega}, c_3^{\omega^2}), (r_3^{\omega^2}, c_3^1) \quad (3.12)$$

One can obtain the following graph displayed in Fig. 1.

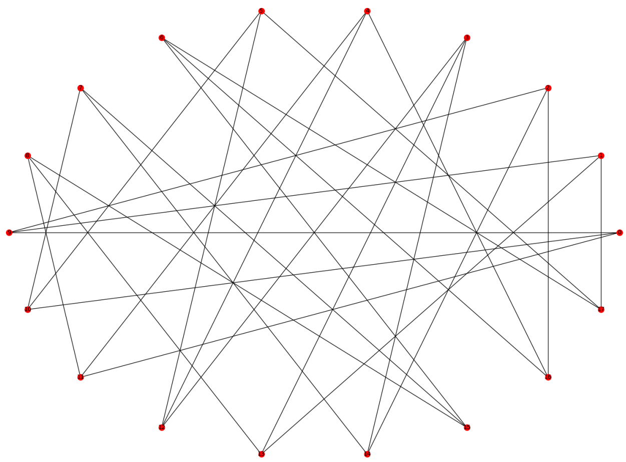

Fig. 1. graph obtained from $GH(3,3)$

From Fig. 1, it can be observed that there are 18 vertices and each vertex has degree 3. Thus, 3− regular graph can be obtained from $GH(3,3)$.

3.1.2 When $n = 2$

This gives the graph for the generalized Hadamard matrix $GH(3,9)$. Some edges can be given as follows:

$$h_{11} = 1 \Rightarrow (r_1^1, c_1^1), (r_1^{\omega}, c_1^{\omega}), (r_1^{\omega^2}, c_1^{\omega^2}) \quad (3.13)$$

$$h_{44} = \omega \Rightarrow (r_4^1, c_4^{\omega}), (r_4^{\omega}, c_4^{\omega^2}), (r_4^{\omega^2}, c_4^1) \quad (3.14)$$

$$h_{75} = \omega^2 \Rightarrow (r_7^1, c_5^{\omega^2}), (r_7^{\omega}, c_5^1), (r_7^{\omega^2}, c_5^{\omega}) \quad (3.15)$$

One can obtain the following graph displayed in Fig. 2.

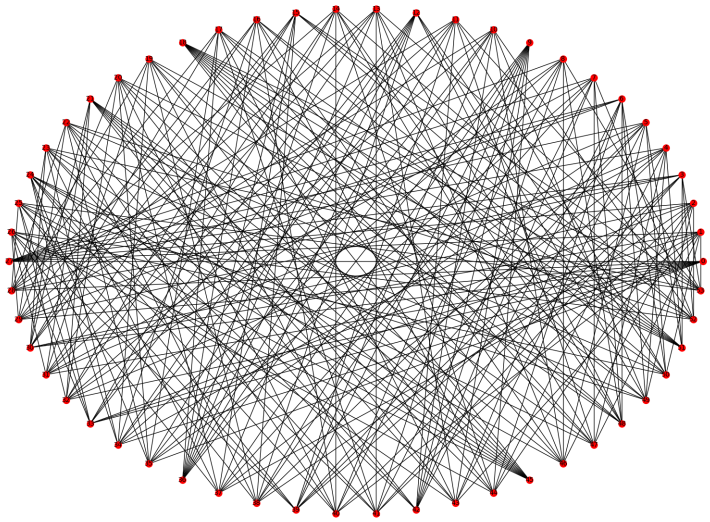

Fig. 2. graph obtained from $GH(3, 9)$

From Fig. 2, it can be seen that the graph has 54 vertices and each vertex has degree 9. Hence, 9− regular graph can be obtained from $GH(3, 9)$

3.2 Case II: Graph obtained from $GH(5, 5^n)$

The Latin square related with $GH(5, 5^n)$ can be shown as follows:

$$A_1 = \begin{bmatrix} 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ \omega & \omega^2 & \omega^3 & \omega^4 & 1 \\ \omega^2 & \omega^3 & \omega^4 & 1 & \omega \\ \omega^3 & \omega^4 & 1 & \omega & \omega^2 \\ \omega^4 & 1 & \omega & \omega^2 & \omega^3 \end{bmatrix} = \begin{bmatrix} L_0 & L_1 & L_2 \end{bmatrix} \quad (3.16)$$

Then 10.5^n symbols $r_i^1, r_i^\omega, r_i^{\omega^2}, r_i^{\omega^3}, r_i^{\omega^4}, c_j^1, c_j^\omega, c_j^{\omega^2}, c_j^{\omega^3}, c_j^{\omega^4}$ for $i, j = 1, 2, \cdots, 5^{n-1}$ would be the vertices of the graph. According to the definition, edges $(r_i^{L_0}, c_j^{L_m})$ if $h_{ij} = \omega^m$, where $m = 0, 1, 2, 3, 4$. Then we have 5^{2n+1} number of edges.

3.2.1 When $n = 1$

This gives the graph for the generalized Hadamard matrix $GH(5, 5)$. Some edges can be given as follows:

$$h_{11} = 1 \Rightarrow (r_1^1, c_1^1), (r_1^\omega, c_1^\omega), (r_1^{\omega^2}, c_1^{\omega^2}), (r_1^{\omega^3}, c_1^{\omega^3}), (r_1^{\omega^4}, c_1^{\omega^4}) \quad (3.17)$$

$$h_{22} = \omega \Rightarrow (r_2^1, c_2^\omega), (r_2^\omega, c_2^{\omega^2}), (r_2^{\omega^2}), c_2^{\omega^3}), (r_2^{\omega^3}, c_2^{\omega^4}), (r_2^{\omega^4}, c_2^1) \quad (3.18)$$

$$h_{34} = \omega^3 \Rightarrow (r_3^1, c_4^{\omega^3}), (r_3^\omega, c_4^{\omega^4}), (r_3^{\omega^2}, c_4^1), (r_3^{\omega^3}, c_4^\omega), (r_3^{\omega^4}, c_4^{\omega^2}) \quad (3.19)$$

One can obtain the following graph displayed in Fig. 3.

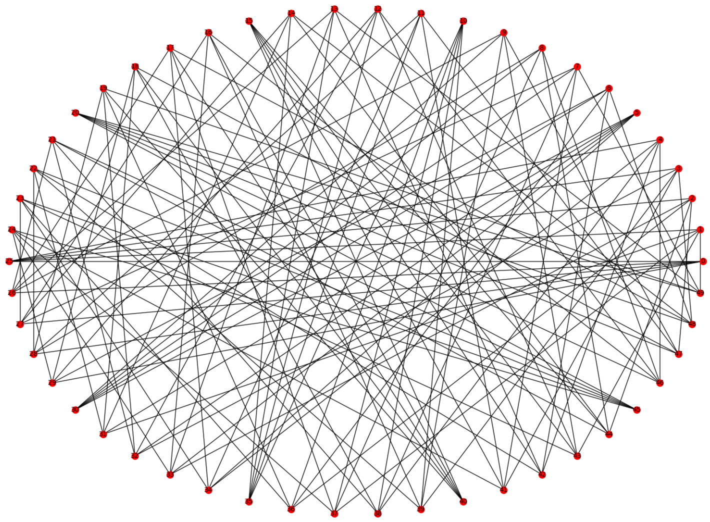

Fig. 3. graph obtained from $GH(5,5)$

From Fig. 3, it can be observed that the graph has 50 vertices and each vertex has degree 5. Hence, 5− regular graph can be obtained from $GH(5,5)$

4 Conclusions

a In section (1), a brief illustration on generalized Hadamard matrices and fundamentals in our research has been discussed.

b In section (2), we have developed a method of constructing generalized Hadamard matrices of prime power orders and verify the method by giving a theorem with proof. In particular, the illustrative examples have been discussed case wise.

c In section (3), we have worked on generating graphs from generalized Hadamard matrices and have developed an algorithm. Further, the algorithm has been automated using Python software. We have identified that the generalized Hadamard graph obtained from generalized Hadamard matrix $GH(p,p^n)$ has a special property, namely p^n- regular. In addition, it has been verified by giving a theorem with proof. Our results have been illustrated by constructing p^n- regular graphs for different values of p and n and this might be used for further identification of properties of generalized Hadamard graphs.

.

Acknowledgment

We thank Mr. H. M. C. A. Herath at the Department of Statistics and Computer Science, University of Peradeniya for useful discussion on Python.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Miyamoto M. A construction of Hadamard matrices. Journal of Combinatorial Theory. Series A. 1991;57:86-108.

[2] Vangelars H, Koukouvinos C, Seberry J. Applications of Hadamard matrices. Journal of Telecommunication and Information Technology; 2003.

[3] Brouwer AE, Cohen AM, Neumaier A. Distance regular graphs. New York: Springer-Verlag. 1989;19-20.

[4] Ito N. Hadamard graphs. I, Graphs and Combinatorics. 1985;1:57-64.

[5] Brock BW, Compton R, De Launey W, Seberry J. On generalized Hadamard matrices and difference matrices:$\mathbf{Z}_6$; 2015.

[6] Horadam KJ. Hadamard matrices and their applications. Princeton: Princeton University Press; 2007.

[7] Craigen R, De Launey W. Generalized Hadamard matrices whose transposes are not generalized Hadamard matrices. In: J. Combinatorial Designs. 2009;17(6):456458.

[8] Butson, A.T. (1962). Generalized Hadamard matrices. Proc. Amer. Math. Soc, 13 , 894898.

[9] De Launey W. Generalized Hadamard matrices which are developed modulo a group. Discrete Mathematics. 1992;104:49-65.

[10] Colbourn CJ, Dinitz JH. Handbook of combinatorial designs (second ed.). Chapman and Hall, CRC, Boca Raton, FL. 2007;301305.

[11] Wallis WD. Certain graphs arising from Hadamard matrices. BULL. AUSTRAL. MATH. SOC. 1969;I:325-331.

[12] Wilson RJ. Introduction to graph theory (Fourth Ed.); 1996.

[13] Nishadi WV, Dhananjaya KDE, Perera AAI. An algorithm to construct symmetric latin squares of order q^n for $q \geq 2$ and $n \geq 1$. American Journal of Engineering Research. 2017;6(2):42-50.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/63334