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Local linear scale factors in map projections in the direction of coordinate axes

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ABSTRACT
This paper explains that the terms "horizontal and vertical scales" are not appropriate in map projections theory. Instead, the authors suggest using the term "scales in the direction of coordinate axes." Since it is not possible to read a local linear scale factor in the direction of a coordinate axis immediately from the definition of a local linear scale factor, this paper considers the derivation of new formulae that enable local linear scale factors in the direction of coordinate x and y axes to be calculated. The formula for computing the local linear scale factor in any direction defined by dx and dy is also derived. Furthermore, the position and magnitude of the extreme values of the local linear scale factor are considered and new formulas derived.

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Map projection; scale factor; horizontal scale factor; vertical scale factor

1. Introduction

Goodchild (1992) notes the importance of map projections and that GIScience needs to revive the orthographic projection to conduct analyses at the global level. Map projections are among the most important topics in Geographic Information System (GIS) curricula as identified in a survey of GIS education in higher education (Fagin and Wikle 2011). Understanding the nuances of map projections and global coordinate systems will become more important as theory and practice in global sciences increase (Greco 2018).

A map is a result of mapping of data usually from the real Earth, celestial body or imagined world to a plane representation on a piece of paper or on a digital display such as a computer monitor. The mapping from the curved surface into a plane is known as map projection and can take a variety of forms. All map projections involve distortion of areas, angles, and/or distances. The types of distortion can be controlled to preserve specific characteristics, but map projections must distort other characteristics of the object represented. The main problem in cartography is that it is not possible to map/project/transform a spherical or ellipsoidal surface into a plane without distortions. Euler first proved as early as 1777 that a sphere cannot be mapped into a plane without distortions (Euler 1777; Biernacki 1949; Lapaine et al. 2014). Gauss's theory applies to the projection of any curved surface on another curved surface, and the Tissot's indicatrix is often used to give a quick and complete information regarding the distortion characteristic in a certain point of the map (Canters and Decleir 1989).

High-resolution regional and global raster databases are often generated for a variety of environmental and scientific modeling applications. The projection of these data from geographic coordinates to a plane coordinate system can result in significant areal distortion. Sources of error include users selecting an inappropriate projection or incorrect parameters for a given projection, algorithm errors in commercial GIS software, and distortions resulting from the projection of data in the raster format. The accuracy of raster projection has been analyzed by Usery and Seong (2001).

Seong and Usery (2001) investigated the effect of projection distortion on raster representation at a global scale and suggested a scale factor model to simulate the effect of using horizontal and vertical scale factors. The results showed that the raster representation accuracy was a function of these two local scale factors. When three global equal-area projections were tested – the cylindrical equal-area, sinusoidal, and Mollweide – the differences between the experimental results and model results were less than 1.0%. Interestingly, the sinusoidal projection showed no error.

According to Seong and Usery (2001), "In the case where the areal scale factor is not 1.0, which occurs with conformal, equidistant, and arbitrary projections, the error will be increased by the extent of the areal expansion. For those non-equal area projections, it is necessary to calculate the vertical and horizontal scale factors independently, because one is not the reciprocal of the other. Because equivalent area is preserved when the product of the horizontal and vertical scale factors equals one, subtracting 1.0 from the multiplication of horizontal and vertical

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scale factors yields the extent of the areal expansion". The authors of the cited paper did not define horizontal and vertical scales in general and did not explain why they argued the area was preserved when the product of the horizontal and vertical scale factors equaled one.

In a footnote on page 224 of the paper by Seong, Mulcahy, and Usery (2002), we read, "In the case of a sphere, the condition for equal area transformation is [$m \cdot n\cos(\varepsilon) = 1.0$], where m is the linear scale factor along the meridian, n is the linear scale factor along the parallel, and ε is the deviation of the graticule intersection from a right angle on the map (Bugayevskiy and Snyder 1995). Because parallels are horizontal in the sinusoidal projection, n becomes the horizontal scale factor, and $m \cdot \cos(\varepsilon)$ produces the vertical scale factor. Here, the vertical scale factor means the scale factor along the right angle on the map, not the scale factor along the meridian. In the case of the sinusoidal projection, n is always 1.0, and m equals $\sec(\varepsilon)$ (Bugayevskiy and Snyder 1995). The vertical scale factor of the sinusoidal projection, therefore, becomes $\sec(\varepsilon)\cdot\cos(\varepsilon)$, which is 1.0. This research focuses on the horizontal and vertical scale factors." It is not clear from the cited footnote why $m\cdot\cos(\varepsilon)$ produces the vertical scale factor, because $m\cdot\cos(\varepsilon)$ is the orthogonal projection of the scale factor along the meridian on a perpendicular to the projection of the parallel.

From the terminological point of view, we see another problem. Namely, *horizontal* and *vertical* are terms often used in the geosciences. For example, vertical is in the direction of a plumbline, while a *vertical scale* is something completely different. Furthermore, if we agree that horizontal and vertical scales are in fact scales in the direction of coordinate axes, then there is another problem: $m\cdot\cos(\varepsilon)$ is not the scale factor of the y-axis. For instance, in a pseudocylindrical projection like the sinusoidal (Sanson) projection, this is an orthogonal projection of the scale along a meridian to the y-axis. But the scale factor in the direction of the y-axis can be obtained when we put $dx = 0$ into the general expression of local linear scale factor. The result will be different from $m\cdot\cos(\varepsilon)$. The proof will be given in the next sections. The proposed approach is valid generally and not only for raster maps.

Moreover, the new formula for computing the local linear scale factor in any direction defined by dx and dy is also derived. Finally, the position and magnitude of the extreme values of the local linear scale factor are considered and new formulas derived. All derived formulas are applied to several map projections to illustrate their functionality and validity. There is no similar research work to this study.

2. A sphere, map projection and local linear scale factor

The geographic parameterization of a sphere with a radius $R > 0$ and the center located in the origin of the coordinate system is a mapping defined by the following formulae:

$$x = R\cos\varphi\cos\lambda,\ y = R\cos\varphi\sin\lambda,\ z = R\sin\varphi \quad (1)$$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$. In this case, φ is latitude, and λ is longitude. It is not difficult to derive that the first fundamental form of this mapping reads

$$ds^2 = R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2 \quad (2)$$

We will limit ourselves to the sphere as a domain of map projection, and define map projection as mapping given by differentiable functions:

$$x = x(\varphi, \lambda),\ y = y(\varphi, \lambda) \quad (3)$$

where the geographic coordinates $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\lambda \in [-\pi, \pi]$, as usual, and x and y are coordinates of a point in a rectangular (mathematical, right oriented) coordinate system in the plane. The first fundamental form of such a mapping is (Canters and Decleir 1989; Bugayevskiy and Snyder 1995):

$$ds'^2 = Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2 \quad (4)$$

with coefficients

$$E = \left(\frac{\partial x}{\partial\varphi}\right)^2 + \left(\frac{\partial y}{\partial\varphi}\right)^2,\ F = \frac{\partial x}{\partial\varphi}\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\varphi}\frac{\partial y}{\partial\lambda},$$
$$G = \left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 \quad (5)$$

The local linear scale factor c for mapping (3) of a sphere with the radius R is usually defined in the theory of map projections by using the following relation:

$$c^2 = \frac{ds'^2}{ds^2} = \frac{Ed\varphi^2 + 2Fd\varphi d\lambda + Gd\lambda^2}{R^2 d\varphi^2 + R^2\cos^2\varphi d\lambda^2} \quad (6)$$

which can also be written as (Canters and Decleir 1989; Bugayevskiy and Snyder 1995)

$$c^2(\alpha) = \frac{E}{R^2}\cos^2\alpha + \frac{F}{R^2\cos\varphi}\sin 2\alpha + \frac{G}{R^2\cos^2\varphi}\sin^2\alpha \quad (7)$$

where

$$\tan\alpha = \frac{\cos\varphi d\lambda}{d\varphi} \quad (8)$$

The poles are singular points of geographic parameterization (1) and therefore expression (6) and all subsequent ones should be interpreted in the poles as limiting cases when $\varphi \to \frac{\pi}{2}$ or $\varphi \to -\frac{\pi}{2}$.

If $\alpha = 0$ or, more generally, $\alpha = z\pi$, $z \in Z$, where Z denotes the set of all integers, then the local linear scale factor c along a meridian ($d\lambda = 0$) is

$$h = c(d\lambda = 0) = \frac{\sqrt{E}}{R} \tag{9}$$

and if $\alpha = \frac{\pi}{2}$ or, more generally, $\alpha = \frac{\pi}{2} + z\pi$, $z \in Z$, then the local linear scale factor c along a parallel $(d\varphi = 0)$ is given by

$$k = c(d\varphi = 0) = \frac{\sqrt{G}}{R\cos\varphi} \tag{10}$$

3. Local linear scale factors in the directions of coordinate axes

It is not possible to read a local linear scale factor in the direction of a coordinate axis immediately from the definition of local linear scale factor (6). The same is true for Equation (7) where α denotes the azimuth, i.e. the angle between a meridian and any direction in a point in question. To be able to get a local linear scale factor in a direction defined by dx and dy we need to modify Equation (6) or (7) in the appropriate way.

Let us start with the general Equation (3) of a map projection. Then we can write

$$dx = \frac{\partial x}{\partial\varphi}d\varphi + \frac{\partial x}{\partial\lambda}d\lambda,\ dy = \frac{\partial y}{\partial\varphi}d\varphi + \frac{\partial y}{\partial\lambda}d\lambda \tag{11}$$

From (11) we have

$$\begin{aligned} d\varphi &= -\frac{1}{H}\left(\frac{\partial y}{\partial\lambda}dx - \frac{\partial x}{\partial\lambda}dy\right), \\ d\lambda &= \frac{1}{H}\left(\frac{\partial y}{\partial\varphi}dx - \frac{\partial x}{\partial\varphi}dy\right) \end{aligned} \tag{12}$$

where

$$H = \sqrt{EG - F^2} = \frac{\partial y}{\partial\varphi}\frac{\partial x}{\partial\lambda} - \frac{\partial y}{\partial\lambda}\frac{\partial x}{\partial\varphi} \tag{13}$$

and

$$H > 0 \tag{14}$$

If we suppose that

$$dy = 0 \tag{15}$$

then

$$d\varphi = -\frac{1}{H}\frac{\partial y}{\partial\lambda}dx,\ d\lambda = \frac{1}{H}\frac{\partial y}{\partial\varphi}dx \tag{16}$$

and by substituting (16) in (6), we get the local linear scale factor in the direction of the x-axis

$$c(dy = 0) = \frac{H}{R\sqrt{\left(\frac{\partial y}{\partial\lambda}\right)^2 + \cos^2\varphi\left(\frac{\partial y}{\partial\varphi}\right)^2}} \tag{17}$$

If we suppose that

$$dx = 0 \tag{18}$$

then

$$d\varphi = \frac{1}{H}\frac{\partial x}{\partial\lambda}dy,\ d\lambda = -\frac{1}{H}\frac{\partial x}{\partial\varphi}dy \tag{19}$$

and by substituting (19) in (6) we get the local linear scale factor in the direction of the y-axis

$$c(dx = 0) = \frac{H}{R\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2 + \cos^2\varphi\left(\frac{\partial x}{\partial\varphi}\right)^2}} \tag{20}$$

Figure 1 represents a general case of Tissot's indicatrix with local linear scale factors along a meridian $c(d\lambda = 0)$, along a parallel $c(d\varphi = 0)$, in the direction of the x-axis $c(dy = 0)$, in the direction of the y-axis $c(dx = 0)$, and the extremal values c_{min} and c_{max}.

4. Local linear scale factor in a given direction

Let us suppose that we need a local linear scale factor in a given direction. If the direction is defined by $d\varphi$ and $d\lambda$ then we can use Equation (6) and our problem will be solved. If the direction is defined by dx and dy then we can use the following procedure. Let us denote (ψ is known as a meridian convergence)

$$\tan\psi = \frac{dy}{dx} \tag{21}$$

Then,

$$\cos\psi dy = \sin\psi dx \tag{22}$$

and by using (12), (6) can be transformed into

$$\begin{aligned} c^2 &= \frac{H^2(dx^2 + dy^2)}{R^2\left[\left(\frac{\partial y}{\partial\lambda}dx - \frac{\partial x}{\partial\lambda}dy\right)^2 + \cos^2\varphi\left(\frac{\partial y}{\partial\varphi}dx - \frac{\partial x}{\partial\varphi}dy\right)^2\right]} \\ &= \frac{H^2}{R^2(a_1\cos^2\psi + a_2\sin\psi\cos\psi + a_3\sin^2\psi)} \end{aligned} \tag{23}$$

where

$$\begin{aligned} a_1 &= \left(\frac{\partial y}{\partial\lambda}\right)^2 + \cos^2\varphi\left(\frac{\partial y}{\partial\varphi}\right)^2 \\ a_2 &= -2\left(\frac{\partial x}{\partial\lambda}\frac{\partial y}{\partial\lambda} + \cos^2\varphi\frac{\partial x}{\partial\varphi}\frac{\partial y}{\partial\varphi}\right) \\ a_3 &= \left(\frac{\partial x}{\partial\lambda}\right)^2 + \cos^2\varphi\left(\frac{\partial x}{\partial\varphi}\right)^2 \end{aligned} \tag{24}$$

It follows that a local linear scale factor c in direction ψ defined by (21) can be calculated by the formula:

$$c = \frac{H}{R\sqrt{a_1\cos^2\psi + a_2\sin\psi\cos\psi + a_3\sin^2\psi}} \tag{25}$$

where the coefficients a_1, a_2 and a_3 are given by (24), and H by (13).

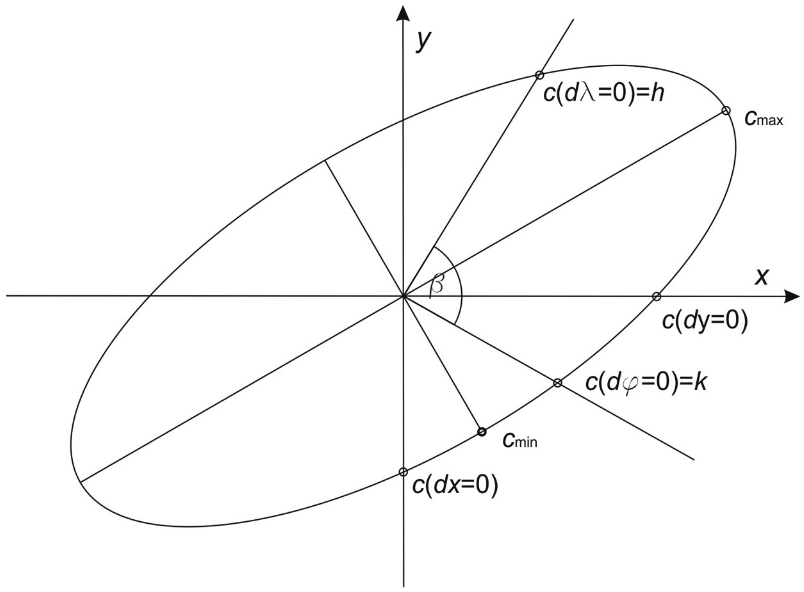

Figure 1. General case of Tissot's indicatrix showing local linear scale factors along a meridian$h = c(d\lambda = 0)$, along a parallel$k = c(d\varphi = 0)$, in the direction of the x-axis $c(dy = 0)$, in the direction of the y-axis $c(dx = 0)$, and extremal values c_{min} and c_{max}. The angle β is the angle between the images of a meridian and a parallel at a point under consideration.

In a special case, when $dy = 0$ then $\psi = 0$, (25) reduces to (17). If $dx = 0$, then $\psi = \frac{\pi}{2}$, and (25) takes the form (20).

Extremal values of $c = c(\psi)$ given by (25) can be obtained in the usual way:

$$\frac{dc}{d\psi} = 0 \tag{26}$$

which gives

$$\tan 2\psi = \frac{a_2}{a_1 - a_3} \tag{27}$$

and by substituting (27) in (25), the extremal values of c are

$$\begin{aligned} c_{1,2} &= \frac{H\sqrt{2}}{R\sqrt{a_1 + a_3 \mp \sqrt{(a_1 - a_3)^2 + a_2^2}}} \\ &= \frac{\sqrt{a_1 + a_3 \pm \sqrt{(a_1 - a_3)^2 + a_2^2}}}{\sqrt{2}R\cos\varphi} \\ &= \frac{\sqrt{G + E\cos^2\varphi \pm \sqrt{(G + E\cos^2\varphi)^2 - 4H^2\cos^2\varphi}}}{\sqrt{2}R\cos\varphi} \\ &= \frac{\sqrt{h^2 + k^2 \pm \sqrt{(h^2 - k^2)^2 + 4h^2k^2\cos^2\beta}}}{\sqrt{2}} \end{aligned} \tag{28}$$

h and k in (28) are local linear scale factors along a meridian and parallel, respectively (see (9) and (10)).

The angle β (Figure 1) is the angle between the images of a meridian and a parallel at a point (Bugayevskiy and Snyder 1995)

$$\sin^2\beta = \frac{H^2}{EG}, \ \cos^2\beta = \frac{F^2}{EG} \tag{29}$$

The extremal values of the local linear scale factor (28) are the semiaxes of the well-known Tissot indicatrix or the ellipse of distortion.

5. Examples

All derived formulas in the previous sections will be applied to several map projections to illustrate their functionality and validity. The first two examples relate to the Mercator projection, which in the normal and transverse aspects is well known and widely used. The following is an example of an azimuth equidistant projection that is chosen because it illustrates the application of formulas to equations in a polar coordinate system. We then investigate pseudocylindrical projections because the sinusoidal (Sanson) projection is one of them. It is this projection that is crucial in our research because we prove that the scale factor in the direction of y-axis of the sinusoidal projection is less than 1 in general. Moreover, in the last example, we prove that an equal-area pseudocylindrical projection with a local linear scale factor in the y-axis direction equal to 1 does not exist.

5.1. Local linear scale factors in the Mercator projection

The equations for the normal aspect conformal cylindrical or Mercator projection of a sphere are

$$x = R(\lambda - \lambda_0),\ y = R\tanh^{-1}\sin\varphi \tag{30}$$

where $\lambda_0 \in [-\pi, \pi]$ represents the longitude of the central meridian. Without loss of generality, we can take $R = 1$. From (30), we can obtain partial derivatives

$$\frac{\partial x}{\partial \lambda} = 1,\ \frac{\partial x}{\partial \varphi} = 0,\ \frac{\partial y}{\partial \lambda} = 0,\ \frac{\partial y}{\partial \varphi} = \frac{1}{\cos\varphi} \tag{31}$$

and then by using formulae (9), (10), (17) and (20)

$$c(d\lambda = 0) = c(d\varphi = 0) = c(dx = 0) = c(dy = 0) = \frac{1}{\cos\varphi} = \cosh y \tag{32}$$

as expected because the Mercator projection is conformal.

5.2. Local linear scale factors in the transverse Mercator projection

The equations for the transverse aspect conformal cylindrical or transverse Mercator projection of a sphere are

$$x = R\tanh^{-1}[\sin(\lambda - \lambda_0)\cos\varphi],\ y = R\tan^{-1}\frac{\tan\varphi}{\cos(\lambda - \lambda_0)} \tag{33}$$

where $\lambda_0 \in [-\pi, \pi]$ represents the longitude of the central meridian. Without loss of generality, we can take $R = 1$. From (33) we can obtain partial derivatives

$$\frac{\partial x}{\partial \lambda} = \frac{\cos\varphi\cos\lambda}{1 - \cos^2\varphi\cos^2\lambda},\ \frac{\partial x}{\partial \varphi} = \frac{\sin\varphi\sin\lambda}{1 - \cos^2\varphi\cos^2\lambda},$$
$$\frac{\partial y}{\partial \lambda} = \frac{\sin\varphi\cos\varphi\sin\lambda}{1 - \cos^2\varphi\cos^2\lambda},\ \frac{\partial y}{\partial \varphi} = \frac{\cos\lambda}{1 - \cos^2\varphi\cos^2\lambda} \tag{34}$$

and then by using formulae (9), (10), (17) and (20)

$$c(d\lambda = 0) = c(d\varphi = 0) = c(dx = 0) = c(dy = 0) = \frac{1}{\sqrt{1 - \cos^2\varphi\cos^2\lambda}} = \cosh x \tag{35}$$

as expected because the transverse Mercator projection is also conformal.

5.3. Local linear scale factors in the azimuthal equidistant projection

The equations for the normal aspect azimuthal equidistant projection of a sphere are

$$x = \rho\sin\theta,\ y = \rho\cos\theta \tag{36}$$

where

$$\rho = R\left(\frac{\pi}{2} - \varphi\right),\ \theta = R(\lambda - \lambda_0) \tag{37}$$

$\lambda_0 \in [-\pi, \pi]$ represents the longitude of the central meridian. Without loss of generality we can take $R = 1$. From (36) and (37) we can obtain partial derivatives

$$\frac{\partial x}{\partial \lambda} = \rho\cos(\lambda - \lambda_0),\ \frac{\partial x}{\partial \varphi} = -\sin(\lambda - \lambda_0),$$
$$\frac{\partial y}{\partial \lambda} = -\rho\sin(\lambda - \lambda_0),\ \frac{\partial y}{\partial \varphi} = -\cos(\lambda - \lambda_0) \tag{38}$$

and then by using formulae (9), (10), (17) and (20)

$$h = c(d\lambda = 0) = 1,\ k = c(d\varphi = 0) = \frac{\rho}{\cos\varphi} \tag{39}$$

$$c(dy = 0) = \frac{\rho}{\sqrt{\rho^2\sin^2\lambda + \cos^2\varphi\cos^2\lambda}},$$
$$c(dx = 0) = \frac{\rho}{\sqrt{\rho^2\cos^2\lambda + \cos^2\varphi\sin^2\lambda}} \tag{40}$$

We can see that the local linear scale factors (39) – (40) are different and depend on the direction.

5.4. Local linear scale factors in pseudocylindrical projections

The equations for normal aspect pseudocylindrical projections are

$$x = x(\varphi, \lambda),\ y = y(\varphi) \tag{41}$$

from where

$$\frac{\partial y}{\partial \varphi} = \frac{dy}{d\varphi},\ \frac{\partial y}{\partial \lambda} = 0 \text{ and } H = \frac{dy}{d\varphi}\frac{\partial x}{\partial \lambda} \tag{42}$$

Now we have

$$c(d\lambda = 0) = h = \frac{\sqrt{E}}{R} = \frac{\sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{dy}{d\varphi}\right)^2}}{R} \tag{43}$$

$$c(d\varphi = 0) = k = \frac{\sqrt{G}}{R\cos\varphi} = \frac{\frac{\partial x}{\partial \lambda}}{R\cos\varphi} \tag{44}$$

$$c(dx = 0) = \frac{\frac{dy}{d\varphi}\frac{\partial x}{\partial \lambda}}{R\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \cos^2\varphi\left(\frac{\partial x}{\partial \varphi}\right)^2}} \tag{45}$$

$$c(dy = 0) = \frac{\frac{\partial x}{\partial \lambda}}{R\cos\varphi} = k = c(d\varphi = 0) \tag{46}$$

5.5. Local linear scale factors in the sinusoidal (Sanson) projection

The equations for normal aspect Sanson projections are

$$x = R(\lambda - \lambda_0)\cos\varphi,\ y = R\varphi \tag{47}$$

from where

$$\frac{\partial y}{\partial \varphi} = \frac{dy}{d\varphi} = R,\ \frac{\partial y}{\partial \lambda} = 0\ H = R^2\cos\varphi \tag{48}$$

According to (43) – (46), we have a local linear scale factor along the meridian

$$c(d\lambda = 0) = h = \frac{\sqrt{E}}{R} = \sqrt{1 + (\lambda - \lambda_0)^2 \sin^2\varphi} \tag{49}$$

a local linear scale factor along the parallel

$$c(d\varphi = 0) = k = \frac{\sqrt{G}}{R\cos\varphi} = 1 \tag{50}$$

a local linear scale factor in the direction of the y-axis

$$c(dx = 0) = \frac{1}{\sqrt{1 + (\lambda - \lambda_0)^2 \sin^2\varphi}} = \frac{1}{h} \le 1 \tag{51}$$

and a local linear scale factor in the direction of the x-axis

$$c(dy = 0) = \frac{\frac{\partial x}{\partial \lambda}}{R\cos\varphi} = 1 = k \tag{52}$$

From expression (51), it can be seen that the local linear scale factor in the direction of the y-axis (vertical scale in terminology used by Seong and Usery (2001) and Seong, Mulcahy, and Usery (2002)) is less than 1 in general. It is equal to one along the equator ($\varphi = 0$) and along the central meridian ($\lambda = \lambda_0$) only. This is contrary to the claims by Seong and Usery (2001) and Seong, Mulcahy, and Usery (2002).

Ellipses of distortion or Tissot's indicatrices in a sinusoidal (Sanson) projection are depicted in Figure 2.

5.6. A pseudocylindrical equal-area projection in which each point has a local linear scale factor in the direction of the y-axis equal to 1 does not exist

The sinusoidal (Sanson) projection is pseudocylindrical and equal-area. We proved in the previous section that for this projection. In general, there is no local linear scale factor in the y-axis direction equal to 1. The question naturally arises about the existence of an equal-area pseudocylindrical projection with a local linear scale factor equal to 1 in the y-axis direction at each point. In this section, we will show that such a map projection does not exist.

Let us assume the opposite and that the equations of a pseudocylindrical projection are (41).

Condition (45) with the requirement that the local linear scale factor equals 1 leads to

$$\frac{\frac{dy}{d\varphi}\frac{\partial x}{\partial \lambda}}{R\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \cos^2\varphi\left(\frac{\partial x}{\partial \varphi}\right)^2}} = 1 \tag{53}$$

The equal-area condition for pseudocylindrical projections reads

$$\frac{dy}{d\varphi}\frac{\partial x}{\partial \lambda} = R^2\cos\varphi \tag{54}$$

Without loss of generality, we can assume that $R = 1$, and then from (53) and (54) we derive this partial differential equation:

$$\left(\frac{\partial x}{\partial \lambda}\right)^2 + \cos^2\varphi\left(\frac{\partial x}{\partial \varphi}\right)^2 = \cos^2\varphi \tag{55}$$

This first order partial differential equation is nonlinear with respect to derivatives and is a special case of general nonlinear equation:

$$F(\varphi, \lambda, x, p, q) = 0 \tag{56}$$

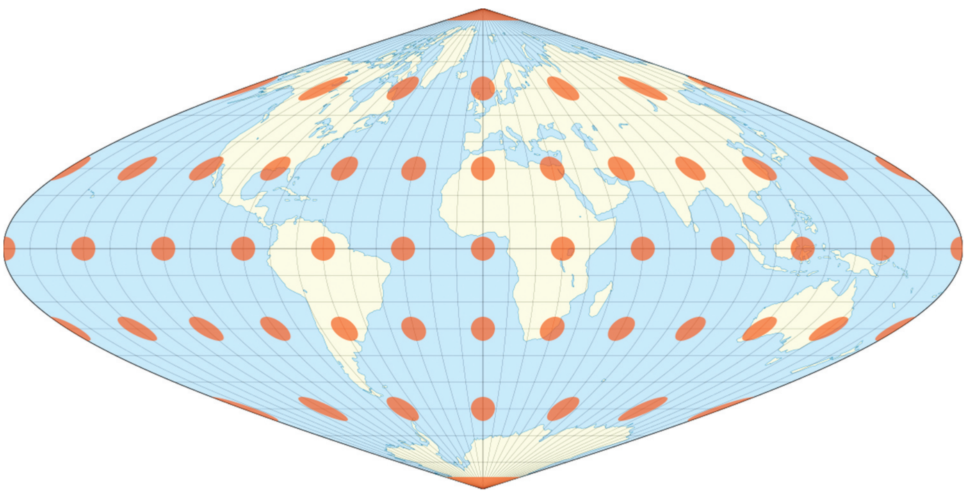

Figure 2. Tissot's indicatrices in a sinusoidal (Sanson) projection (Jung 2020).

where $p = \frac{\partial x}{\partial \varphi}$ and $q = \frac{\partial x}{\partial \lambda}$. The theory of such equations is well known (Sneddon 1957). Its *complete integral* could be written as

$$x(\varphi, \lambda) = a\lambda + b + \int \sqrt{\frac{\cos^2\varphi - a^2}{\cos^2\varphi}} d\varphi \qquad (57)$$

where a and b are real numbers (Polyanin, Zaitsev, and Moussiaux 2002). The expression under the square root in (57) should be a real number for each $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. It follows $a = 0$, and the conclusion: x is not a function of φ and λ, but of φ only. But this is the opposite of the hypothesis. In other words, there is no pseudocylindrical projection that satisfies (55) and has the equation of the form (57).

We conclude that there is no equal-area pseudocylindrical projection which in each of its points has local linear scale factor in the y-axis direction equal to 1.

6. Conclusions

All map projections involve distortion of areas, angles, and/or distances. The types of distortion can be controlled to preserve specific characteristics, but map projections must distort other characteristics of the object represented. The main problem in cartography is that it is not possible to map/project/transform a spherical or ellipsoidal surface into a plane without distortions. It is well known that scale changes from point to point, and at certain points usually depends on direction. This is the local scale. The local linear scale factor c is the ratio of the differential of the curve arc in the plane of projection and the differential of the corresponding curve arc on the ellipsoid or spherical surface. The local linear scale factor c is one of the most important indicators of distortion distribution in the theory of map projections.

It is not possible to read the local linear scale factor in a direction of a coordinate axis immediately from the definition (6). The same is true for Equation (7) where α is the angle between the meridian and any direction in a point in question. In this paper, we derive new formulae that enable calculation of a local linear scale factor in the direction of coordinate axes x and y. Moreover, we derive the formula for computing the local linear scale factor in any direction defined by dx and dy.

The paper proposes avoiding terms like *horizontal and vertical scales* and replacing them with the expression *scales in the direction of coordinate axes*. Furthermore, it was shown that the local linear scale factor in the direction of the y-axis (*vertical scale* in terminology by Seong and Usery (2001) and Seong, Mulcahy, and Usery (2002)) in a sinusoidal (Sanson) projection is less than or equal to 1. It is equal to one along the equator ($\varphi = 0$) and along the central meridian ($\lambda = \lambda_0$) only.

Furthermore, it is shown that there is no pseudocylindrical equal-are projection that would have a local linear scale factor equal to 1 at each point in the y-axis direction.

Finally, the position and magnitude of the extreme values of the local linear scale factor were considered and new formulas were derived.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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Miljenko Lapaine graduated from the Faculty of Science, University of Zagreb, in the field of Theoretical Mathematics. He received his PhD from the Faculty of Geodesy of the University of Zagreb with a dissertation titled *Mapping in the Theory of Map Projections*. He has been a professor emeritus since 2018 and has published 1000 papers. Now he is the Vice-Chair of the Commission on Map Projections of the International Cartographic Association, the President of the Croatian Cartographic Society and the executive editor of the *Cartography and Geoinformation* journal.

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References

Biernacki, F. 1949. *Teoria odwzorowań powierzchni dla geodetów i kartografów*: Warsaw, Główny Urząd Pomiarów Kraju, Prace Geodezyjnego Instytutu Naukowo-Badawczego, no. 4. Polish. Translated into English as *Theory of representation of surfaces for surveyors and cartographers*: U.S. Dept. of Commerce, 1965.

Bugayevskiy, L. M., and J. P. Snyder. 1995. *Map Projections: A Reference Manual*. London: Taylor and Francis.

Canters, F., and H. Decleir. 1989. *The World in Perspective, a Directory of World Map Projections*. Chichester: John Wiley & Sons.

Euler, L. 1777. "De Repraesentatione Superficiei Sphaericae Super Plano", *Acta Academiae Scientiarum Imperialis Petropolitanae*. Translated into German In: "Drei Abhandlungen uber Kartenprojection", *Ostwald's Klassiker der exakten Wissenschaften*, no. 93, pp. 3–37, Leipzig, Wilhelm Engelmann, 1898. Translated into Polish in: Biernacki 1949. Translated into Russian: "Ob izobrazhenii poverhnosti shara na ploskosti", *Trudy Petersburgskoy akademii nauk za 1777 g.*, Tom 1, pp. - 107–132, in: L. Ejler: *Izbrannye kartograficheskhie stat'i*, Tri stat'i po matematicheskhoj kartografii, Izdatel'stvo geodezicheskhoj literatury, 1959, Moscow. Translated into English in: Biernacki 1965. Translated into Croatian in: Lapaine 2014.

Fagin, T. D., and T. A. Wikle. 2011. "The Instructor Element of GIS Instruction at US Colleges and Universities." *Transactions in GIS* 15 (1): 1–15. doi:10.1111/j.1467-9671.2010.01238.x.

Goodchild, M. F. 1992. "Geographical Information Science." *International Journal of Geographic Information Systems* 6 (1): 31–45. doi:10.1080/02693799208901893.

Greco, S. E. 2018. "Seven Possible States of Geospatial Data with respect to Map Projection and Definition: A Novel Pedagogical Device for GIS Education." *Geo-Spatial Information Science* 21 (4): 288–293. doi:10.1080/10095020.2018.1536406.

Jung, T. 2020. "Compare Map Projections." Accessed June 29, 2020. https://map-projections.net/

Lapaine, M. 2014. "Kartografske projekcije i njihove deformacij." [Map Projections and Their Distortions]. In *Proceedings of the Fifth Croatian Congress on Cadastre*, edited by Damir Medak, Milan Rezo, and Mladen Zrinjski, 15–32. Zagreb: Croatian Geodetic Society.

Polyanin, A. D., V. F. Zaitsev, and A. Moussiaux. 2002. *Handbook of First Order Partial Differential Equations*. London: Taylor and Francis.

Seong, J. C., K. A. Mulcahy, and E. L. Usery. 2002. "The Sinusoidal Projection: A New Importance in Relation to Global Image Data." *The Professional Geographer* 54 (2): 218–225. doi:10.1111/0033-0124.00327.

Seong, J. C., and E. L. Usery. 2001. "Assessing Raster Representation Accuracy Using a Scale Factor Model." *Photogrammetric Engineering and Remote Sensing* 67 (10): 1185–1191.

Sneddon, T. A. 1957. *Elements of Partial Differential Equations*. New York: McGraw Hill.

Usery, E. L., and J. C. Seong. 2001. "All Equal-Area Map Projections Are Created Equal, But Some Are More Equal Than Others." *Cartography and Geographic Information Science* 28 (3): 183–194. doi:10.1559/152304001782153053.