British Journal of Mathematics & Computer Science
12(2): 1-7, 2016, Article no.BJMCS.20823
ISSN: 2231-0851

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The Sum and Product of Chromatic Numbers of Graphs and their Line Graphs

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Article Information
DOI: 10.9734/BJMCS/2016/20823
Editor(s):
(1) Metin Baarir, Department of Mathematics, Sakarya University, Turkey.
Reviewers:
(1) Anonymous, Lagos State University, Lagos, Nigeria.
(2) Krasimir Yankov Yordzhev, South-West University, USA.
Complete Peer review History: http://sciencedomain.org/review-history/11710

Original Research Article

Received: 11 August 2015
Accepted: 12 September 2015
Published: 07 October 2015

Abstract

The bounds on the sum and product of chromatic numbers of a graph and its complement are known as Nordhaus-Gaddum inequalities. In this paper, some variations on this result is studies. First, recall their theorem, which gives bounds on the sum and the product of the chromatic number of a graph with that of its complement. We also provide a new characterization of the certain graph classes.

Keywords: Chromatic number of a graph; chromatic index of a graph; line graph.

2010 Mathematics Subject Classification: 05C15.

1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [1],[2],[3]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.

A graph G can be considered as a pair $(V(G), E(G))$ where $V(G)$ is the set of vertices and $E(G)$ is the set of edges of G. The degree of a vertex v of a graph G is the number of edges incident on v and is denoted by $d(v)$. The minimum among the degrees of all the vertices of G is denoted by $\delta(G)$ and the maximum among the degrees of all the vertices of G is denoted by $\Delta(G)$.

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Many problems in extremal graph theory seek the extreme values of graph parameters on families of graphs. The classic paper of Nordhaus and Gaddum [4] study the extreme values of the sum (or product) of a parameter on a graph and its complement, following solving these problems for the chromatic number on n-vertex graphs. In this paper, we study such problems for some graphs and their associated graphs.

Definition 1.1. [3] A *coloring* of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The set of all vertices with any one color is independent and is called a color class. An n-*coloring* of a graph G uses n colors; it thereby partitions V into n color classes. The *chromatic number* $\chi(G)$ is defined as the minimum n for which G has an n - coloring. A graph G is n-*colorable* if $\chi(G) \leq n$ and is n-*chromatic* if $\chi(G) = n$.

Definition 1.2. [3] An *edge-coloring* or *line-coloring* of a graph G is an assignment of colors to its edges (lines) so that no two adjacent edges (lines) are assigned the same color. An n-*edge-coloring* of G is an edge-coloring of G which uses exactly n colors. The *edge-chromatic number* $\chi'(G)$ is the minimum n for which G has an n-edge-coloring.

Recall the following theorem, which gives bounds on the sum and the product of the chromatic number of a graph with that of its complement.

Theorem 1.1. [4] If G is a graph with $V(G) = n$ and chromatic number $\chi(G)$ then

$$2\sqrt{n} \leq \chi(G) + \chi(\bar{G}) \leq n + 1 \tag{1.1}$$

$$n \leq \chi(G).\chi(\bar{G}) \leq \frac{(n+1)^2}{4} \tag{1.2}$$

And there is no possible improvement of any of these bounds. In fact, much more can be said. Let n be a positive integer. For every two positive integers a and b,

$$2\sqrt{n} \leq a + b \leq n + 1 \tag{1.3}$$

$$n \leq ab \leq \frac{(n+1)^2}{4} \tag{1.4}$$

There is a graph G of order n such that $\chi(G) = a$ and $\chi(\bar{G}) = b$.

Definition 1.3. [5] The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G.

Definition 1.4. [6] The *chromatic index* (or *edge chromatic number*) $\chi'(G)$ of a graph G is the minimum positive integer k for which G is $k-$edge colorable. Furthermore, $\chi'(G) = \chi(L(G))$ for every nonempty graph G.

Theorem 1.2. [3] For any graph G, the edge-chromatic number satisfies the inequalities

$$\Delta \leq \chi' \leq \Delta + 1 \tag{1.5}$$

Theorem 1.3. [6] (Konig's Theorem) If G is a nonempty bipartite graph, then $\chi'(G) = \Delta(G)$.

Theorem 1.4. [7] Let $G = K_n$, the complete graph on n vertices, $n \geq 2$. Then

$$\chi'(G) = \begin{cases} \Delta(G) & \textit{if } n \textit{ is even} \\ \Delta(G) + 1 & \textit{if } n \textit{ is odd} \end{cases}$$

We denote the chromatic number of a graph G is denoted by $\chi(G)$ and the complement of G is denoted by $\bar{G}$.

2 New Results

With the above background, we now prove the following.

Proposition 2.1. *For a complete graph K_n, $n \geq 2$,*

$$\chi(K_n) + \chi(L(K_n)) = \begin{cases} 2n-1 & \text{if } n \text{ is even} \\ 2n & \text{if } n \text{ is odd} \end{cases}$$

$$\chi(K_n).\chi(L(K_n)) = \begin{cases} n(n-1) & \text{if } n \text{ is even} \\ n^2 & \text{if } n \text{ is odd} \end{cases}$$

Proof. We know that $\chi(K_n) = n$ for all positive integer n. Let $L(K_n)$ denotes the line graph of K_n. Then, K_n is $(n-1)$ regular and by definition, $\chi(L(K_n)) = \chi'(K_n)$. By theorem 1.4,

$$\chi'(K_n) = \begin{cases} \Delta = n-1 & \text{if } n \text{ is even} \\ \Delta + 1 = n & \text{if } n \text{ is odd} \end{cases}$$

ie,

$$\chi(L(K_n)) = \chi'(K_n) = \begin{cases} \Delta = n-1 & \text{if } n \text{ is even} \\ \Delta + 1 = n & \text{if } n \text{ is odd} \end{cases}$$

Therefore

$$\chi(K_n) + \chi(L(K_n)) = \begin{cases} n + n - 1 = 2n - 1 & \text{if } n \text{ is even} \\ n + n = 2n & \text{if } n \text{ is odd} \end{cases}$$

Similarly,

$$\chi(K_n).\chi(L(K_n)) = \begin{cases} n(n-1) & \text{if } n \text{ is even} \\ n.n = n^2 & \text{if } n \text{ is odd} \end{cases}$$

□

Proposition 2.2. *For a complete bipartite graph $K_{m,n}$, $m, n \geq 0$,*

$$\chi(K_{m,n}) + \chi(L(K_{m,n})) = 2 + max\ (m,n) \text{ and}$$
$$\chi(K_{m,n}).\chi(L(K_{m,n})) = 2\ max\ (m,n)$$

Proof. We know that $\chi(K_{m,n}) = 2$ for all positive integer m, n. Let $L(K_{m,n})$ denotes the line graph of $K_{m,n}$. Then, by definition, $\chi(L(K_{m,n})) = \chi'(K_{m,n})$.

Therefore by theorem 1.3, $\chi(L(K_{m,n})) = \chi'(K_{m,n}) = max\ (m,n)$.

Then $\chi(K_{m,n}) + \chi(L(K_{m,n})) = 2 + max\ (m,n)$ and
$\chi(K_{m,n}).\chi(L(K_{m,n})) = 2\ max\ (m,n)$ □

Corollary 2.1. *For a star graph $K_{1,n}$,*

$$\chi(K_{1,n}) + \chi(L(K_{1,n})) = n + 2 \text{ and}$$
$$\chi(K_{1,n}).\chi(L(K_{1,n})) = 2n$$

Proof. Since any two edges of a star graph $K_{1,n}$ are adjacent each other, then its line graph is a complete graph with n vertices. We know $\chi(K_{1,n}) = 2$ and $\chi(L(K_{1,n})) = n$.
Therefore $\chi(K_{1,n}) + \chi(L(K_{1,n})) = n + 2$ and
$\chi(K_{1,n}).\chi(L(K_{1,n})) = 2n$. □

A *bistar graph* $(B_{m,n})$ is a graph obtained by attaching m pendent edges to one end point and n pendent edges to the other end point of K_2.

The following result establishes the sum and product of chromatic numbers of a bistar graph and its line graph.

Proposition 2.3. *For a bistar graph $B_{m,n}$,*

$$\chi(B_{m,n}) + \chi(L(B_{m,n})) = 2 + max\ (m, n) \text{ and}$$
$$\chi(B_{m,n}).\chi(L(B_{m,n})) = 2\ max\ (m, n)$$

Proof. Let u, v be two vertices of K_2. Let m edges be attached to u and n edges be attached to v. Since all m edges at u are adjacent to each other and all n edges at v are adjacent to each other, its line graph is the one point union of 2 complete graphs K_m and K_n.
Then

$$\chi(L(B_{m,n})) = \begin{cases} m & \text{if } m > n \\ n & \text{otherwise} \end{cases}$$

Therefore

$$\chi(B_{m,n}) + \chi(L(B_{m,n})) = 2 + max\ (m, n) \text{ and}$$
$$\chi(B_{m,n}).\chi(L(B_{m,n})) = 2\ max\ (m, n)$$

□

Proposition 2.4. *Let G be a bipartite graph with a bipartition (X, Y) with $|X| = m$ and $|Y| = n$, then $4 \leq \chi(G) + \chi(L(G) \leq 2 + max\ (m, n)$ and*
$4 \leq \chi(G).\chi(L(G) \leq 2\ max\ (m, n)$

Proof. The minimal connected bipartite graph of m, n vertices will be P_{m+n-1} and that of its line graph is P_{m+n-2}. Chromatic number of G and $L(G)$ is 2.

Therefore $\chi(G) + \chi(L(G)) = 4$ and $\chi(G).\chi(L(G)) = 4$ then

$$4 \leq \chi(G) + \chi(L(G)) \leq 2 + max\ (m, n) \tag{2.1}$$

$$4 \leq \chi(G).\chi(L(G)) \leq 2\ max\ (m, n) \tag{2.2}$$

□

Definition 2.1. [3] For $n \geq 4$, a *wheel graph* W_n is defined to be the graph $K_1 + C_{n-1}$, where C_{n-1} is a cycle on $n - 1$ vertices.

Theorem 2.2. *The chromatic index of a wheel graph W_n with n vertices is $n - 1$.*

Proof. A wheel graph W_n with n vertices is $K_1 + C_{n-1}$. Suppose K_1 lies inside the circle C_{n-1}. Let $e_1, e_2, e_3, ..., e_{n-1}$ be the edges incident with the vertex K_1 and we need $n - 1$ colors to color this $n - 1$ edges. Let $u_1, u_2, u_3, ..., u_{n-1}$ be the end vertices of $e_1, e_2, e_3, ..., e_{n-1}$, which form the cycle C_{n-1}. Then, there exists $q_1, q_2, q_3, ..., q_{n-1}$ edges incident to $u_1, u_2, u_3, ..., u_{n-1}$. For any edge q_j in the cycle C_{n-1}, there exists an edge e_i which is not adjacent to q_j. Therefore e_i and q_j can have the same color. That is using the same set of $n - 1$ colors, we can color the edges $q_1, q_2, q_3, ..., q_{n-1}$. That means we can color the edges of a wheel graph W_n with $n - 1$ colors or the chromatic index of W_n is $n - 1$. □

Proposition 2.5. *For a wheel graph W_n on n vertices and $2(n - 1)$ edges, $n \geq 4$,*

$$\chi(W_n) + \chi(L(W_n)) = \begin{cases} n + 3 & \text{if } n \text{ is even} \\ n + 2 & \text{if } n \text{ is odd} \end{cases}$$

$$\chi(W_n).\chi(L(W_n)) = \begin{cases} 4(n - 1) & \text{if } n \text{ is even} \\ 3(n - 1) & \text{if } n \text{ is odd} \end{cases}$$

Proof. We know that

$$\chi(W_n) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

for all positive integer $n \geq 4$. Let $L(W_n)$ denotes the line graph of W_n. Then, $\chi(L(W_n)) = \chi'(W_n) = (n-1)$ Therefore

$$\chi(W_n) + \chi(L(W_n)) = \begin{cases} 4 + (n-1) = n+3 & \text{if } n \text{ is even} \\ 3 + (n-1) = n+2 & \text{if } n \text{ is odd} \end{cases}$$

Similarly,

$$\chi(W_n).\chi(L(W_n)) = \begin{cases} 4(n-1) & \text{if } n \text{ is even} \\ 3(n-1) & \text{if } n \text{ is odd} \end{cases}$$

□

Definition 2.2. [8] *Helm graphs* are graphs obtained from a wheel by attaching one pendant edge to each vertex of the cycle.

Theorem 2.3. The chromatic index of a helm graph H_n with $2n+1$ vertices and $3n$ edges is n.

Proof. Let u_1 is the central vertex and $v_1, v_2, v_3, ..., v_n$ be the vertices of the cycle. Let $w_1, w_2, w_3, ..., w_n$ be the pendent vertices attached to $v_1, v_2, v_3, ..., v_n$ respectively. Let $e_1, e_2, e_3, ..., e_n$ be the edges incident on the vertex u_1. Let $l_1, l_2, l_3, ..., l_n$ be the edges of the cycle formed by the vertices $v_1, v_2, v_3, ..., v_n$. Let $q_1, q_2, q_3, ..., q_n$ be the pendent edges. Since each $e_1, e_2, e_3, ..., e_n$ are adjacent to each other, to color the edges $e_1, e_2, e_3, ..., e_n$, we need at least n colors. For every edge l_i, we can find at least one edge e_j such that $l_i and e_j$ are non-adjacent. Color the edge l_i with the same color of e_j. Using the same set of n colors, we can color all the edges $e_1, e_2, e_3, ..., e_n$ and $l_1, l_2, l_3, ..., l_n$. For any edge q_k, there will be at least one edge e_j and at least one edge l_i with the same color and are non-adjacent to q_k. Now assign this color to q_k. Hence we color all the vertices of helm graph using the same set of n colors. □

Proposition 2.6. *For a helm graph H_n on $2n+1$ vertices, and $3n$ edges, $n \geq 3$,*

$$\chi(H_n) + \chi(L(H_n)) = \begin{cases} n+4 & \textit{if } n \textit{ is even} \\ n+3 & \textit{if } n \textit{ is odd} \end{cases}$$

$$\chi(H_n).\chi(L(H_n)) = \begin{cases} 4n & \textit{if } n \textit{ is even} \\ 3n & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. We know that

$$\chi(H_n) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

for all positive integer $n \geq 4$. Let $L(H_n)$ denotes the line graph of H_n. Then, $\chi(L(H_n)) = \chi'(H_n) = n$

Therefore

$$\chi(H_n) + \chi(L(H_n)) = \begin{cases} n+4 & \text{if } n \text{ is even} \\ n+3 & \text{if } n \text{ is odd} \end{cases}$$

Similarly,

$$\chi(H_n).\chi(L(H_n)) = \begin{cases} 4n & \text{if } n \text{ is even} \\ 3n & \text{if } n \text{ is odd} \end{cases}$$

□

Definition 2.3. [5] Given a vertex x and a set U of vertices, an $x, U-$fan is a set of paths from x to U such that any two of them share only the vertex x.

Theorem 2.4. *The chromatic index of a fan graph $F_{1,n}$ with $n+1$ vertices is n.*

Proof. The fan graph $F_{1,n}$ with $n+1$ vertices is $K_1 + P_{n-1}$, where P_{n-1} is a path on $n-1$ vertices. Let $e_1, e_2, e_3, ..., e_n$ be the edges incident with the vertex K_1 and we need n colors to color this n edges. Let $q_1, q_2, q_3, ..., q_{n-1}$ be the edges in the path P_{n-1}. For any edge q_j in the path P_{n-1}, there exists an edge e_i which is not adjacent to q_j. Therefore e_i and q_j can have the same color. That is, by taking $(n-1)$ colors out of n colors, we can color the edges $q_1, q_2, q_3, ..., q_{n-1}$. That is we can color the edges of a fan graph $F_{1,n}$ with n colors or the chromatic index of $F_{1,n}$ is n. □

Proposition 2.7. *For a fan graph $F_{1,n}$,*

$$\chi(F_{1,n}) + \chi(L(F_{1,n})) = n + 4 \text{ and}$$
$$\chi(F_{1,n}).\chi(L(F_{1,n})) = 3(n+1)$$

Proof. For a fan graph $F_{1,n}$, with $e \geq 1$, we have $\chi(F_{1,n}) = 3$ for all positive integer $n \geq 2$. Let $L(F_{1,n})$ denotes the line graph of $F_{1,n}$. Then $\chi(L(F_{1,n})) = \chi'(F_{1,n}) = n$.
Therefore $\chi(F_{1,n}) + \chi(L(F_{1,n})) = 3 + n = n + 3$ and
$\chi(F_{1,n}).\chi(L(F_{1,n})) = 3n$. □

3 Conclusions

The theoretical results obtained in this research may provide a better insight into the problems involving chromatic number by improving the known lower and upper bounds on sums and products of chromatic numbers of a graph G and an associated graph of G. More properties and characteristics of operations on chromatic number and also other graph parameters are yet to be investigated. The problems of establishing the inequalities on sums and products of chromatic numbers for various graphs and graph classes still remain unsettled. All these facts highlight a wide scope for further studies in this area.

Acknowledgement

This work is motivated by the inspiring talk given by Dr. J Paulraj Joseph, Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli - 627012, TamilNadu, Titled **Bounds on sum of graph parameters - A survey**, at the National Conference on Emerging Trends in Graph Connections (NCETGC-2014), on January 8-10, 2014, at the Department of Mathematics, University of Kerala, Kariavattom, Kerala.

Competing Interests

The authors declare that no competing interests exist.

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