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Bounds for the Blow-up Time and Blow-up Rate Estimates for Nonlinear Parabolic Equations with Dirichlet or Neumann Boundary Conditions

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Abstract

This paper is concerned with the blow-up phenomena for a type of parabolic equations with weighted nonlinear source

$$\begin{cases} (b(u))_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x,u), & x \in \Omega, t > 0, \\ u(x,t) = 0 \text{ or } \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = g(x) \geq 0, & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N (N \geq 3)$ is a smooth bounded domain. Through constructing some suitable auxiliary functions and using the first-order differential inequality technique, we obtain the bounds for the blow-up time and the estimates of the blow-up rate of the solution to the problem.

Keywords: Parabolic equation; bounds of blow-up time; estimates of blow-up rate; weighted nonlinear source.

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1 Introduction and the Main Results

In this paper, we are concerned with the blow-up phenomenon of the following problem:

$$\begin{cases} (b(u))_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(x,u), & x \in \Omega, t > 0, \\ u(x,t) = 0 \text{ or } \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = g(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial}{\partial n}$ represents the outward normal derivative on $\partial\Omega$, $g(x)$ is the initial value, $1 < p \leq 2$. Set $R^+ := (0, +\infty)$. We assume, throughout the work, that (F1):$f(x,s)$ is a nonnegative $C^1(\overline{\Omega} \times [0, +\infty))$ function, and (F2):$\int_s^{+\infty} \frac{d\eta}{f(\cdot,\eta)}$ is bounded for $s > 0$, b is a $C^2(R^+)$ function satisfying $1 \leq b'_m \leq b'(s) \leq b'_M, b''(s) \leq 0$ for all $s > 0$.

The phenomena of the blow-up for nonlinear parabolic equations have been investigated extensively by many authors (see [1-7] and the referen-ces therein). Some special cases of (1.1) have been studied already, such as model problem (1.2) which often occurs in many mathematical models of applied science, such as chemical reactions, heat transfer, population dynamic and electro-rheological fluids(see [8,9] and the references therein). There are many topics of interest on these models, for example, the conditi-ons of blow-up and global existence of the solution etc, we refer the reader to [10,11,12-14,15] and the references therein. In many situations, the techniques used in the study of blow-up phenomena lead to the bounds for the blow-up time when blow-up occurs. Payne, Schaefer [16] obtained the lower bounds for blow-up time in parabolic problems under Neumann boundary conditions. Later, many authors got the bounds for the blow-up time of the solution to some models (see [17,18,19,20] and the references therein). In applications, the lower bound seems to be more important, due to the explosive nature of the solution. And there are many results about this aspect, we can refer [21],[22],[23-26] and the references therein. Many approa-ches have been developed in discussing the bounds for the blow-up time of the solution to various parabolic problems. However, the blow-up rate of the solution to the problem with general nonlinearity is unknown. K.Baghaei, M. B. Ghaemi and M. Hesaaraki [27] studied the following semi-linear parabolic problem with a variable source:

$$\begin{cases} u_t = \Delta u + u^{p(x)}, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbf{R}^N (N \geq 3)$ is a bounded domain with smooth boundary. A lower bound for the blow-up time was obtained when blow-up occurs for $1 < p_- \leq p_+ < +\infty$. Wu, Guo, Gao [28] got an upper bound for the blow-up time of the solution to (1.2) by constructing a new control function and applying suitable embedding theorems. Under the conditions $1 < p_- \leq p_+ \leq \frac{n+2}{n-2}$, certain initial data and positive initial energy, Wang, He [29] also obtained an upper bound for the blow-up time of the solution to (1.2). The following problem was investigated by Song and Lv in [30]:

$$\begin{cases} u_t = \Delta u + a(x)f(u), & x \in \Omega, t > 0, \\ u(x,t) = 0 \text{ or } \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = g(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.3}$$

which often appears in combustion theories and engineering application. In the model, $\Omega \subset \mathbf{R}^N (N \geq 3)$ is a smooth bounded domain. By using the first-order differential inequality technique, the bounds for blow-up time of the solution to (1.3) were obtained. In addition, the estimates of the blow-up rate were also obtained. In some special case, the authors got the exact values of blow-up rate and blow-up time.

In this paper, through constructing auxiliary functions $A(t)$, $B(t)$ and using the first-order differential inequality technique, we investigate the bounds for blow-up time and blow-up rate of the solution to the problem (1.1). In section 2, the lower bounds for the blow-up time and blow-up rate of the solution to (1.1) are specified under different assumptions on the function f. In section 3, the upper bounds for blow-up time and blow-up rate are obtained under some appropriate assumptions on the functions f and g. A remark and some examples will be given in section 4. Our results extend and supplement those obtained in [27-29]-[30].

The following conditions will be required in our results:

(F3) There exist positive constants $C_1,\ C_2,\ M,\ k$, a nonnegative constant r and a positive function $m(x) \in C(\Omega; R^+)$ satisfy $0 \le r \le 1$, $\frac{1-r}{M} < m_- := \inf_{x\in\Omega} m(x) \le m(x) \le m_+ := \sup_{x\in\Omega} m(x) \le k+1$ such that

$$f(x,s) \le C_1 + C_2 s^r \Big(\int_\Omega s^{m(x)} dx\Big)^M,\ for\ all\ s \ge 0;$$

(F4) There exist positive constants $C_3,\ C_4,\ k$ and a positive function $m(x) \in C(\Omega; R^+)$ satisfy $\frac{3}{4} < m_- \le m(x) \le m_+ < \infty$, $k > \max\{(n-1)(4m_+ - 3), 1\}$ such that

$$f(x,s) \le C_3 + C_4 s^{m(x)};$$

(F5) There exist positive constant α such that

$$sf(x,s) \ge 2(1+\alpha)F(x,s),$$

where $F(x,s) = \int_0^s f(x,\zeta)d\zeta$;

(G1) For $1 < p \le 2$,

$$\int_\Omega |\nabla g|^p dx \le p \int_\Omega F(x,g)dx.$$

2 Lower Bounds for the Blow-up Time of the Solution

In this section, we will study the lower bound for the blow-up time and blow-up rate of the solution to (1.1) under different assumptions on the function f. The following auxiliary functions are used:

$$G(s) = (k+1)\int_0^s \eta^k b'(\eta)d\eta,\quad A(t) = \int_\Omega G(u(x,t))dx, \tag{2.1}$$

where k is a positive constant.

Theorem 2.1. Let u be a nonnegative solution of (1.1) subject to Dirichlet (or Neumann) boundary condition, $A(t)$ be defined as (2.1). Assume that f satisfies (F1), (F2) and (F3), then the blow-up time t^* is bounded from below by

$$t^* \ge \int_{A(0)}^{+\infty} \frac{d\eta}{K_1\eta^{r_1} + K_2\eta^{r_2}(1+\eta^{r_3})^M}.$$

Moreover, we have the following blow-up rate estimate

$$\|u(\cdot,t)\|_{L^{k+1}} \ge S_1^{\frac{1}{k+1}}(t^*-t)^{-\frac{1}{r+m_+M-1}},$$

where $K_1,\ K_2,\ r_1,\ r_2,\ r_3$ and S_1 are positive constants which will be determined later.

Proof. Applying the divergence theorem and taking into account assumption (F3), we obtain

$$\begin{aligned} A'(t) &= \int_\Omega G'(u(x,t))u_t dx \\ &= (k+1)\int_\Omega u^k b'(u)u_t dx \\ &= (k+1)\int_\Omega u^k[\mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(x,u)]dx \\ &= -k(k+1)\int_\Omega u^{k-1}|\nabla u|^p dx + (k+1)\int_\Omega u^k f(x,u)dx \\ &\le C_1(k+1)\int_\Omega u^k dx + C_2(k+1)\int_\Omega u^{k+r}dx(\int_\Omega u^{m(x)}dx)^M. \end{aligned} \tag{2.2}$$

For each $t > 0$, we divide Ω into two sets,

$$\Omega_{\{<1\}} = \{x \in \Omega : u(x,t) < 1\}, \ \ \Omega_{\{\ge 1\}} = \{x \in \Omega : u(x,t) \ge 1\}.$$

Now, applying Hölder inequality, we have

$$\int_\Omega u^{k+r}dx \le |\Omega|^{\frac{1-r}{k+1}}(\int_\Omega u^{k+1}dx)^{\frac{k+r}{k+1}}, \tag{2.3}$$

and

$$\begin{aligned} \int_\Omega u^{m(x)}dx &\le \int_{\Omega_{\{<1\}}} u^{m_-}dx + \int_{\Omega_{\{\ge 1\}}} u^{m_+}dx \\ &\le \int_\Omega u^{m_-}dx + \int_\Omega u^{m_+}dx \\ &\le (\int_\Omega u^{k+1}dx)^{\frac{m_-}{k+1}}|\Omega|^{1-\frac{m_-}{k+1}} + (\int_\Omega u^{k+1}dx)^{\frac{m_+}{k+1}}|\Omega|^{1-\frac{m_+}{k+1}}. \end{aligned} \tag{2.4}$$

Substituting (2.3), (2.4) into (2.2), we obtain

$$\begin{aligned} A'(t) &\le C_1(k+1)|\Omega|^{\frac{1}{k+1}}(\int_\Omega u^{k+1}dx)^{\frac{k}{k+1}} + C_2(k+1)|\Omega|^{\frac{1-r}{k+1}}(\int_\Omega u^{k+1}dx)^{\frac{k+r}{k+1}} \\ &\quad [(\int_\Omega u^{k+1}dx)^{\frac{m_-}{k+1}}|\Omega|^{1-\frac{m_-}{k+1}} + (\int_\Omega u^{k+1}dx)^{\frac{m_+}{k+1}}|\Omega|^{1-\frac{m_+}{k+1}}]^M \\ &\le K_1(\int_\Omega u^{k+1}dx)^{\frac{k}{k+1}} + K_2(\int_\Omega u^{k+1}dx)^{\frac{k+r+m_-M}{k+1}}[1 + (\int_\Omega u^{k+1}dx)^{\frac{m_+-m_-}{k+1}}]^M, \end{aligned} \tag{2.5}$$

where

$K_1 = C_1(k+1)|\Omega|^{\frac{1}{k+1}}$,
$K_2 = C_2(k+1)|\Omega|^{\frac{1-r}{k+1}} max\{|\Omega|^{\frac{M(k+1-m_-)}{k+1}}, |\Omega|^{\frac{M(k+1-m_+)}{k+1}}\}$.

On the other hand,

$$A(t) = \int_\Omega G(u(x,t))dx \ge \int_\Omega u^{k+1}dx, \tag{2.6}$$

combing with (2.5), we have

$$A'(t) \le K_1(A(t))^{\frac{k}{k+1}} + K_2(A(t))^{\frac{k+r+m_-M}{k+1}}[1 + (A(t))^{\frac{m_+-m_-}{k+1}}]^M. \tag{2.7}$$

Integrating (2.7) from 0 to t ($t < t^*$), if $\lim_{t\to t^*} A(t) = +\infty$, we get

$$t^* \ge \int_{A(0)}^{+\infty} \frac{d\eta}{K_1\eta^{r_1} + K_2\eta^{r_2}(1+\eta^{r_3})^M}, \tag{2.8}$$

where $r_1 = \frac{k}{k+1}$, $r_2 = \frac{k+r+m_-M}{k+1}$, $r_3 = \frac{m_+-m_-}{k+1}$.

Integrating (2.7) from t to t^*, we obtain

$$t^* - t \ge \int_{A(t)}^{\infty} \frac{d\eta}{K_1\eta^{r_1} + K_2\eta^{r_2}(1+\eta^{r_3})^M} = \phi(A)(t), \tag{2.9}$$

obviously, $\phi(A)(t)$ is a decreasing function of A which means its inverse function ϕ^{-1} exists and it is also a decreasing one. Therefore, we have

$$A(t) \geq \phi^{-1}(t^* - t), \tag{2.10}$$

which gives the lower estimate of blow-up rate. In fact, if t close to t^* enough, such that

$$K_2 \eta^{\frac{k+r+m_+M}{K+1}} > K_1 \eta^{r_1},$$

using (2.9), we have

$$t^* - t \geq \frac{k+1}{2K_2(r+m_+M-1)}(A(t))^{\frac{1-r-m_+M}{k+1}}, \tag{2.11}$$

which means that

$$A(t) \geq (\frac{k+1}{2K_2(r+m_+M-1)})^{\frac{k+1}{r+m_+M-1}}(t^*-t)^{-\frac{k+1}{r+m_+M-1}}. \tag{2.12}$$

Since $A(t) \leq b'_M \int_\Omega u^{k+1}dx$, combing with (2.12), we have

$$\|u(\cdot,t)\|_{L^{k+1}} \geq S_1^{\frac{1}{k+1}}(t^*-t)^{-\frac{1}{r+m_+M-1}}, \tag{2.13}$$

where $S_1 = \frac{1}{b'_M}[\frac{k+1}{2K_2(r+m_+M-1)}]^{\frac{k+1}{r+m_+M-1}}$.

Remark. This method is valid for $1 < p < \infty$ and not to restrict the space dimension.

Theorem 2.2. Let u be a nonnegative solution of (1.1) subject to Dirichlet boundary condition, $A(t)$ be defined as (2.1). Assume that f satisfies the conditions (F1), (F2) and (F4), then the blow-up time t^* is bounded from below. We have

$$t^* \geq \int_{A(0)}^{+\infty} \frac{d\eta}{K_3 + K_4\eta^{\frac{k}{k+1}} + K_5\eta^{\frac{3(n-p)}{3n-4p}}},$$

and blow-up rate estimate

$$\|u(\cdot,t)\|_{L^{k+1}} \geq S_2^{\frac{1}{k+1}}(t^*-t)^{-\frac{3n-4p}{p(k+1)}},$$

where K_3, K_4, K_5 and S_2 are positive constants which will be defined later.

Proof. From (2.2) and (F4), we know that

$$\begin{aligned} A'(t) &= -k(k+1)\textstyle\int_\Omega u^{k-1}|\nabla u|^p dx + (k+1)\int_\Omega u^k f(x,u)dx \\ &\leq -k(k+1)(\tfrac{p}{k-1+p})^p \textstyle\int_\Omega |\nabla u^{\frac{k-1+p}{p}}|^p dx + C_3(k+1)\int_\Omega u^k dx \\ &\quad + C_4(k+1)\textstyle\int_\Omega u^{k+m(x)}dx. \end{aligned} \tag{2.14}$$

Like (2.4),

$$\int_\Omega u^{k+m(x)}dx \leq \int_\Omega u^{k+m_-}dx + \int_\Omega u^{k+m_+}dx, \tag{2.15}$$

by applying Hölder inequality, we have

$$\int_\Omega u^{k+m_-}dx \leq |\Omega|^{m_1}(\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}}dx)^{m_2}, \tag{2.16}$$

and

$$\int_\Omega u^{k+m_+}dx \leq |\Omega|^{m_3}(\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}}dx)^{m_4}, \tag{2.17}$$

where

$$m_1 = 1 - \frac{4(n-p)(k+m_-)}{k(4n-3p)+p(n-3)+2n},\ m_2 = \frac{4(n-p)(k+m_-)}{k(4n-3p)+p(n-3)+2n},$$
$$m_3 = 1 - \frac{4(n-p)(k+m_+)}{k(4n-3p)+p(n-3)+2n},\ m_4 = \frac{4(n-p)(k+m_+)}{k(4n-3p)+p(n-3)+2n}.$$

Substituting (2.16), (2.17) into (2.15) and using Young inequality, we get

$$\int_\Omega u^{k+m(x)}dx \le l_1 + l_2 \int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}}dx, \tag{2.18}$$

where $l_1 = (m_1+m_3)|\Omega|$, $l_2 = m_2 + m_4$. Substituting (2.18) into (2.14), we have

$$\begin{aligned} A'(t) \quad &\le -k(k+1)(\tfrac{p}{k-1+p})^p \textstyle\int_\Omega |\nabla u^{\frac{k-1+p}{p}}|^p dx + C_3(k+1)|\Omega|^{\frac{1}{k+1}}(\int_\Omega u^{k+1}dx)^{\frac{k}{k+1}} \\ &+C_4 l_1(k+1) + C_4 l_2(k+1)\textstyle\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}}dx. \end{aligned} \tag{2.19}$$

We now make use of Hölder inequality to the last term on the right side of (2.19) to get

$$\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}}dx \le (\int_\Omega u^{k+1}dx)^{\frac{3}{4}}(\int_\Omega (u^{\frac{k-1+p}{p}})^{\frac{np}{n-p}}dx)^{\frac{1}{4}}. \tag{2.20}$$

Note that

$$\int_\Omega (u^{\frac{k-1+p}{p}})^{\frac{np}{n-p}}dx \le (C_S)^{\frac{np}{n-p}}(\int_\Omega (|\nabla u^{\frac{k-1+p}{p}}|^p dx)^{\frac{n}{n-p}}, \tag{2.21}$$

here C_s is the best Sobolev's constant. By inserting (2.21) in (2.20) and using the Young inequality, we have

$$\begin{aligned} \textstyle\int_\Omega u^{\frac{k(4n-3p)+p(n-3)+2n}{4(n-p)}}dx \quad &\le \tfrac{(3n-4p)(C_s)^{\frac{np}{4(n-p)}}}{4(n-p)\epsilon^{\frac{n}{3n-4p}}}(\textstyle\int_\Omega u^{k+1}dx)^{\frac{3(n-p)}{3n-4p}} \\ &+\tfrac{n\epsilon(C_s)^{\frac{np}{4(n-p)}}}{4(n-p)}\textstyle\int_\Omega |\nabla u^{\frac{k-1+p}{p}}|^p dx, \end{aligned} \tag{2.22}$$

where ϵ is a positive constant to be determined later. Combing with (2.22) and (2.19), we obtain

$$\begin{aligned} A'(t) \quad &\le K_3 + K_4(\textstyle\int_\Omega u^{k+1}dx)^{\frac{k}{k+1}} + K_5(\int_\Omega u^{k+1}dx)^{\frac{3(n-p)}{3n-4p}} + K_6\int_\Omega |\nabla u^{\frac{k-1+p}{p}}|^p dx \\ &\le K_3 + K_4(A(t))^{\frac{k}{k+1}} + K_5(A(t))^{\frac{3(n-p)}{3n-4p}} + K_6\textstyle\int_\Omega |\nabla u^{\frac{k-1+p}{p}}|^p dx, \end{aligned} \tag{2.23}$$

where

$$K_3 = C_4 l_1(k+1),\ K_4 = C_3(k+1)|\Omega|^{\frac{1}{k+1}},\ K_5 = C_4 l_2(k+1)\frac{(3n-4p)(C_s)^{\frac{np}{4(n-p)}}}{4(n-p)\epsilon^{\frac{n}{3n-4p}}},$$

$K_6 = C_4 l_2(k+1)\frac{n\epsilon(C_s)^{\frac{np}{4(n-p)}}}{4(n-p)} - k(k+1)(\frac{p}{k-1+p})^p.$

If we choose $\epsilon > 0$ such that

$\epsilon = \frac{4k(n-p)(\frac{p}{k-1+p})^p}{C_4 l_2 n (C_s)^{\frac{np}{4(n-p)}}},$

then, we obtain the differential inequality

$$A'(t) \le K_3 + K_4(A(t))^{\frac{k}{k+1}} + K_5(A(t))^{\frac{3(n-p)}{3n-4p}}. \tag{2.24}$$

An integrating of the differential inequality (2.24) from 0 to t $(t < t^*)$ leads to

$$t^* \geq \int_{A(0)}^{\infty} \frac{d\eta}{K_3 + K_4\eta^{\frac{k}{k+1}} + K_5\eta^{\frac{3(n-p)}{3n-4p}}}, \tag{2.25}$$

if $lim_{t\to t^*} A(t) = +\infty$. Similar to (2.13), we get the lower estimate of the blow-up rate

$$\|u(\cdot,t)\|_{L^{k+1}} \geq S_2^{\frac{1}{k+1}}(t^*-t)^{-\frac{3n-4p}{p(k+1)}}, \tag{2.26}$$

where $S_2 = \frac{3n-4p}{2b'_M K_5 P}$.

3 Upper Bounds for the Blow-up Time of the Solution

In this section we seek the upper bound for the blow-up time and corresponding estimates of the blow-up rate. Define

$$B(t) = 2\int_\Omega \int_0^u sb'(s)dsdx,\ H(t) = \int_\Omega (F(x,u) - \frac{1}{p}|\nabla u|^p)dx. \tag{3.1}$$

Then, we have the following results.

Theorem 3.1. Let u be a nonnegative solution of (1.1) subject to Dirichlet (or Neumann) boundary condition, $B(t)$, $H(t)$ be defined as (3.1). Assume that f satisfies (F1), (F2) and (F5), then the blow-up time t^* is bounded from up. We have

$$t^* \leq \frac{B(0)}{4\alpha(1+\alpha)H(0)},$$

and blow-up rate estimate

$$\|u(\cdot,t)\|_{L^2} \leq S_3^{\frac{1}{2}}(t^*-t)^{-\frac{\alpha}{2}},$$

where $S_3 = \frac{(B(0))^{1+\alpha}}{4\alpha(1+\alpha)H(0)b'_m}$.

Proof. Multiplying the first equality of (1.1) by u_t and integrating the resulting equality over Ω, we have

$$\begin{aligned} \textstyle\int_\Omega (b(u))_t u_t dx &= \textstyle\int_\Omega \operatorname{div}(|\nabla u|^{p-2}\nabla u)u_t dx + \int_\Omega f(x,u)u_t dx \\ &= -\textstyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla u_t dx + \frac{d}{dt}(\int_\Omega F(x,u)dx) \\ &= -\textstyle\frac{d}{dt}(\frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x,u)dx) \\ &= H'(t). \end{aligned} \tag{3.2}$$

From condition (G1) and the fact that $\int_\Omega (b(u))_t u_t dx \geq 0$, we know that $H'(t) \geq 0$, $H(t) \geq 0$. Next, we compute

$$\begin{aligned} B'(t) &= 2\textstyle\int_\Omega ub'(u)u_t dx \\ &= 2\textstyle\int_\Omega u[\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x,u)]dx \\ &\geq -2\textstyle\int_\Omega |\nabla u|^p dx + 4(1+\alpha)\int_\Omega F(x,u)dx \\ &\geq 4(1+\alpha)\textstyle\int_\Omega [F(x,u) - \frac{1}{p}|\nabla u|^p]dx \\ &= 4(1+\alpha)H(t), \end{aligned} \tag{3.3}$$

it follows that

$$(1+\alpha)B'(t)H(t) \leq \frac{1}{4}(B'(t))^2 = (\int_\Omega ub'(u)u_t dx)^2 \leq (\int_\Omega u^2 b'(u)dx)(\int_\Omega b'(u)u_t^2 dx). \tag{3.4}$$

By integrating by parts on $\int_0^u sb'(s)ds$, it is easy to see that $\int_\Omega u^2b'(u)dx \le B(t)$, then

$$(1+\alpha)B'(t)H(t) \le B(t)H'(t), \tag{3.5}$$

which means that

$$\frac{d}{dt}[(B(t))^{-(1+\alpha)}H(t)] \ge 0, \tag{3.6}$$

Integrating (3.6) from 0 to t, we have

$$(B(t))^{-(1+\alpha)}H(t) \ge (B(0))^{-(1+\alpha)}H(0), \tag{3.7}$$

that is

$$\frac{H(t)}{H(0)} \ge (\frac{B(t)}{B(0)})^{1+\alpha}. \tag{3.8}$$

Combing with (3.3), we get

$$\frac{B'(t)}{(B(t))^{1+\alpha}} \ge \frac{4(1+\alpha)H(0)}{(B(0))^{1+\alpha}} = l_3. \tag{3.9}$$

Integrating (3.9) from 0 to t, we have

$$(B(t))^{-\alpha} \le (B(0))^{-\alpha} - \alpha l_3 t. \tag{3.10}$$

Since inequality (3.10) cannot hold for $(B(0))^{-\alpha} - \alpha l_3 t \le 0$, we thus conclude that solution u of the problem (1.1) blows up at some finite time t^* and t^* is bounded from up by

$$t^* \le \frac{(B(0))^{-\alpha}}{\alpha l_3} = \frac{B(0)}{4\alpha(1+\alpha)H(0)}. \tag{3.11}$$

Integrating (3.11) from t to t^*, we have

$$B(t) \le \frac{(B(0))^{1+\alpha}}{4\alpha(1+\alpha)H(0)}(t^*-t)^{-\alpha}. \tag{3.12}$$

Since $B(t) \ge b'_m \int_\Omega u^2dx$, from (3.12), we get

$$\|u(\cdot,t)\|_{L^2} \le S_3^{\frac{1}{2}}(t^*-t)^{-\frac{\alpha}{2}}, \tag{3.13}$$

where $S_3 = \frac{(B(0))^{1+\alpha}}{4\alpha(1+\alpha)H(0)b'_m}$.

4 A Remark and Some Examples

In this section we will give a remark and two examples to illustrate the results in our work and make some discussions.

Remark 4.1. When $b(u) \equiv u$, $p = 2$, $f(x,u) = u^{m(x)} = u^{p(x)}$, the problem (1.1) under Dirichlet boundary condition is model (1.2). Moreover, if m_- and k in Theorem 2.2 satisfy $m_- > 1$, $k > \max\{2(n-2)(m_+-1), 1\}$, we can obtain similar result in [27]. That is, blow-up time is bounded from below by

$$\int_{A(0)}^{+\infty} \frac{d\eta}{K_7 + K_8\eta^{\frac{3(n-2)}{3n-8}}}, \tag{4.1}$$

here $K_7 = (m_1 + m_3)|\Omega|(k + 1)$, $K_8 = \frac{(k+1)(C_s)^{\frac{n}{2(n-2)}}(3n-8)(m_2+m_4)}{4(n-2)\epsilon^{\frac{n}{3n-8}}}$. Besides, we have the corresponding estimate of blow-up rate

$$\|u\|_{L^{K+1}} \geq (\frac{3n-8}{4K_8})^{\frac{1}{k+1}}(t^* - t)^{-\frac{3n-8}{2(k+1)}}. \tag{4.2}$$

If the positive constants α in (F5) satisfies $0 < \alpha < \frac{m_- - 1}{2}$ $(m_- = p_- > 1)$, by the results of Theorem 3.1, we get the upper bound for the blow-up time and the estimate of blow-up rate, that is

$$t^* \leq \frac{B(0)}{4\alpha(1+\alpha)H(0)}, \tag{4.3}$$

and

$$\|u\|_{L^2} \leq \frac{(B(0))^{1+\alpha}}{4\alpha(1+\alpha)H(0)}(t^* - t)^{-\frac{\alpha}{2}}, \tag{4.4}$$

with $B(0) = \int_\Omega g^2(x)dx$, $H(0) = \int_\Omega (F(x,g) - \frac{1}{2}|\nabla g|^2)dx$.

Example 4.2. Let us consider the following example:

$$\begin{cases} (u + \sqrt{1+u})_t = \text{div}(|\nabla u|^{p-2}\nabla u) + (\int_\Omega u^\beta dx)^{\frac{q}{\beta}}, & x \in \Omega, t > 0, \\ u(x,t) = 0 \text{ or } \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = g(x) \geq 0, & x \in \Omega, \end{cases} \tag{4.5}$$

where $1 < p \leq 2$, $0 < \beta \leq k+1$, $q > 1$. Now we have

$$b(u) = u + \sqrt{1+u}, \quad f(x,u) = (\int_\Omega u^\beta dx)^{\frac{q}{\beta}}.$$

Applying Theorem 2.1, we have

$$t^* \geq \frac{1}{K_9}(A(0))^{\frac{1-q}{k+1}}, \tag{4.6}$$

and

$$\|u\|_{L^{k+1}} \geq (\frac{1}{2K_9})^{\frac{k+1}{q-1}}(t^* - t)^{-\frac{k+1}{q-1}}, \tag{4.7}$$

where $K_9 = (q-1)|\Omega|^{\frac{k+2-\beta}{k+1}}$.

Applying Theorem 3.1, we have

$$t^* \leq \frac{B(0)}{4\alpha(1+\alpha)H(0)}, \tag{4.8}$$

and

$$\|u\|_{L^2} \leq S_4^{\frac{1}{2}}(t^* - t)^{-\frac{\alpha}{2}}, \tag{4.9}$$

where $S_4 = \frac{(B(0))^{1+\alpha}}{4\alpha(1+\alpha)H(0)}$.

From (4.6) and (4.8), we get the bounds for the blow-up time of the solution to the problem (4.5), that is

$$\frac{1}{K_9}A(0)^{\frac{1-q}{k+1}} \leq t^* \leq \frac{B(0)}{4\alpha(1+\alpha)H(0)}. \tag{4.10}$$

Example 4.3. Let u be a nonnegative solution of the following problem:

$$\begin{cases} (u + 2\sqrt{1+u})_t = \text{div}(|\nabla u|^{p-2}\nabla u) + u^{2+\sum_{i=1}^N x_i^2}, & x \in \Omega, t > 0, \\ u(x,t) = 0 & x \in \partial\Omega, t > 0, \\ u(x,0) = 4 - \sum_{i=1}^N x_i^2, & x \in \Omega, \end{cases} \tag{4.11}$$

where $\Omega = \{x = (x_1, x_2, ..., x_N)| \sum_{i=1}^{N} x_i^2 < 1\}$ is a unit ball of $\mathbf{R}^N$, $1 < p \leq 2$. Now we have

$$b(u) = u + 2\sqrt{1+u},\ f(x,u) = u^{1+\sum_{i=1}^{N} x_i^2},\ g(x) = 4 - \sum_{i=1}^{N} x_i^2.$$

Substituting $m_- = 2$, $m_+ = 3$, $c_3 = 0$, $c_4 = 1$, $b'_M = 2$ into Theorem 2.2, we could get the lower bound for the blow-up time and blow-up rate of the solution to the problem (4.11) like (2.25), (2.26). From Theorem 3.1, we know that

$$t^* \leq \frac{B(0)}{4\alpha(1+\alpha)H(0)}, \tag{4.12}$$

and

$$\|u\|_{L^2} \leq (\frac{(B(0))^{1+\alpha}}{4\alpha(1+\alpha)H(0)})^{\frac{1}{2}}(t^* - t)^{-\frac{\alpha}{2}}. \tag{4.13}$$

As

$$B(0) = 2\int_\Omega \int_0^g sb'(s)dsdx \leq 4\int_\Omega \int_0^4 sdsdx = 32|\Omega|, \tag{4.14}$$

and

$$\begin{aligned} H(0) &= \int_\Omega (F(x,g) - \tfrac{1}{p}|\nabla g|^p)dx \\ &= \int_\Omega (\tfrac{g^{3+\sum_{i=1}^N x_i^2}}{3+\sum_{i=1}^N x_i^2} - \tfrac{2^p}{p}(\textstyle\sum_{i=1}^N x_i^2)^{\frac{p}{2}})dx \\ &\geq \tfrac{27p-2^{p+2}}{4p}|\Omega|. \end{aligned} \tag{4.15}$$

Taking (4.14) and (4.15) into (4.12), (4.13), we obtain

$$t^* \leq \frac{32p}{\alpha(1+\alpha)(27p-2^{p+2})}, \tag{4.16}$$

and

$$\|u\|_{L^2} \leq (\frac{32p|\Omega|^\alpha}{\alpha(1+\alpha)(27p-2^{p+2})})^{\frac{1}{2}}(t^* - t)^{-\frac{\alpha}{2}}. \tag{4.17}$$

5 Conclusion

We construct some suitable auxiliary functions and find the upper and lower bounds for the blow-up time when blow-up phenomena occurs, accordingly, give the estimates of the blow-up rate of the solution to the problem.

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Competing Interests

The authors declare that no competing interests exist.

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